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GAUSS QUADRATURE RULES INVOLVING SOME NONCLASSICAL WEIGHT FUNCTIONS

by Paul F. Byrd and David C. Galant

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SYMBOLS

b_n	parameter in recurrence relation (see eq. (15))
$e_k^{(n)}$	modified differences in Q-D scheme
E_N	error term of a quadrature formula (see eq. (2))
g_n	parameter in recurrence relation (see eq. (15))
k_N	coefficient in the error term (see eq. (3))
M_n	nth moment of weight function $w(t)$ (see eq. (11))
N	number of nodes in quadrature rule (see eq. (1))
$q_k^{(n)}$	modified quotients in Q-D scheme
S	significant figures
t_{jN}	$j = 1(1)N$, nodal points in quadrature formula (see eq. (1))
$w(t)$	weight function
W_{jN}	$j = 1(1)N$, weight coefficients in quadrature rule (see eq. (1))
$\Gamma(z)$	gamma function of argument z
$\phi_k(t)$	orthogonal polynomial of degree k (see eq. (15))

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SUMMARY

Gauss quadrature rules are given for numerically evaluating integrals on the interval [0, 1] for some weight functions of the form $w(t) = t^{\gamma}(1 - t^{\alpha})^{\beta}$, where α is a positive integer ≥ 2 and $\beta, \gamma > -1$. The nodes, weight coefficients, and the parameters in the three-term recurrence relation for the sequence of orthogonal polynomials associated with $w(t)$ are tabulated to 25 significant figures for N-point rules with $N = 2, 4, 6, 8, 12, 16$, and 24. Numerical tables to 5S are also given for a coefficient in the expression of the error term.

INTRODUCTION

A wide variety of problems requiring numerical approximations to specific definite integrals arise very often in practice, usually as an intermediate step in a more extensive computation. There are many methods for approximating such integrals, but among all quadrature formulas of the form

$$I = \int_a^b g(t)dt \equiv \int_a^b w(t)f(t)dt \simeq \sum_{j=1}^{N} W_{jN} f(t_{jN}) \tag{1}$$

only the weighted gaussian types have all their parameters chosen to maximize performance. For these formulas, the nodes t_{jN} and the coefficients W_{jN} are uniquely determined so that all approximations to the given integral are exact for arbitrary polynomials f of the highest possible degree, $2N - 1$. If the function f is other than a polynomial of degree $\leq 2N - 1$, the integration formulas (1) have an error term given (e.g., see ref. 1) by

$$E_N = \frac{k_N}{(2N)!} f^{(2N)}(\tau) , \qquad a < \tau < b \tag{2}$$

where

$$k_N = \int_a^b w(t)\phi_N^2(t)dt \tag{3}$$

with ϕ_N being the associated Nth-degree orthogonal polynomial.

The literature concerning weighted gaussian formulas is very large, both classical and recent (refs. 2, 3, and 4). Widespread use of high-speed computers, with emphasis on economizing the number of operations required in computations, has resulted in considerable renewed interest in the power of the gaussian technique; gauss quadrature rules are being tabulated and used with increasing frequency. Very accurate rules, for example, were recently tabulated in references 1 and 5 for cases associated with the classical orthogonal polynomials of Legendre, Laguerre, Chebyshev, Jacobi, and Hermite. Recent numerical tables to many significant figures have also been given for some nonclassical cases such as a weight function on $[0, 1]$ having logarithmic singularities (ref. 6) and the weight function e^{-x^2} on the interval $[0, \infty]$ (ref. 7).

Weighted gaussian integration formulas are excellent for large classes of functions arising in practice. They are particularly useful when a complete integrand $g(t)$ is factorable into two functions $f(t)$ and $w(t)$, where f is well approximated by a polynomial, with the nonnegative weight function w having a singularity or a singular derivative at one or both ends of the interval. Since neither the error term (2) nor the sum in the approximations (1) directly involves the weight function w, once the quadrature rules are constructed and programmed on a computer, they can be repeatedly applied to integrals of the form (1) with different functions $f(t)$ and the same $w(t)$. There are several other advantages of gauss-type formulas: (a) since all the coefficients W_{jN} turn out to be positive, the rules have good roundoff properties; (b) the sequence of formulas (1) for $N = 1, 2, \ldots$ will always converge to the value of the Riemann integral I whenever this integral exists; (c) the rules have more favorable error terms and generally achieve higher accuracy with the use of fewer nodes than Newton-Cotes methods.

This report presents tabulated gaussian quadrature formulas for

$$I(\alpha, \beta, \gamma; f) = \int_0^1 w(\alpha, \beta, \gamma; t) f(t) dt \approx \sum_{j=1}^{N} W_{jN} f(t_{jN}) \tag{4}$$

where the weight function w is of the form

$$w(\alpha, \beta, \gamma; t) \equiv w(t) = t^{\gamma}(1 - t^{\alpha})^{\beta} \tag{5}$$

with α as a positive integer ≥ 2; $\beta, \gamma > -1$; and w or its derivative singular at one or both ends of the interval. For the special case $\alpha = 1$, the approximate relation (4) becomes the classical Gauss-Jacobi formula

$$\int_0^1 t^{\gamma}(1 - t)^{\beta} f(t) dt \approx \sum_{j=1}^{N} W_{jN} f(t_{jN}) \tag{6}$$

Krylov, Lugin, and Janovic (ref. 8) recently gave these quadrature rules to eight significant figures for $\beta, \gamma = -0.9(0.1)3.0$, $\beta \leqslant \gamma$ and $N = 1(1)8$.

We shall consider various cases where the parameters (α, β, γ) are

$$\left.\begin{aligned} \alpha &= 2, 3, 4, 6, 8 \\ \beta &= -3/4, -2/3, -1/2, 1/2 \\ \gamma &= -1/2, 0, 1/2 \end{aligned}\right\} \tag{7}$$

One of these, for example, is the integral

$$I(3, -1/2, 0; f) = \int_0^1 \frac{f(t)}{\sqrt{1 - t^3}}\, dt \tag{8}$$

which, when $f(t)$ is simply a rational function, may be reduced by a substitution involving Jacobian elliptic functions to a combination of three standard tabulated elliptic integrals (ref. 9). A similar statement applies to integrals $I(\alpha, \beta, \gamma; f)$ with regard to the other values α, β, γ given in equations (7).

If f is not a rational function (e.g., e^{-ct}), none of the integrals $I(\alpha, \beta, \gamma)$ are expressible in finite terms of known tabulated functions for the chosen parameters (α, β, γ), and the only recourse is numerical approximation via some quadrature formula. Even when such integrals can be written as combinations of standard elliptic integrals etc., it is probably more efficient to use quadrature rules having the highest degree of algebraic precision to obtain answers. Integrals with our weight functions $w(t)$ occur sufficiently frequently in practice to warrant the construction of such rules.

Numerical tables for N-point gaussian rules $\{W_{jN}, t_{jN}\}$ with $N = 2, 4, 6, 8, 12, 16$, and 24 are given to 25 significant figures for 12 special cases of $I(\alpha, \beta, \gamma; f)$. These are for

$$(\alpha, \beta, \gamma) = \left\{\begin{matrix} (3, -1/2, 0) & (3, 1/2, 0) & (4, -1/2, 0) & (4, 1/2, 0) \\ (6, -1/2, 0) & (6, 1/2, 0) & (8, -1/2, 0) & (8, 1/2, 0) \\ (3, -1/2, -1/2) & (3, 1/2, 1/2) & (2, -3/4, 0) & (2, -2/3, 0) \end{matrix}\right\} \tag{9}$$

The coefficients b_n, g_n in the three-term recursion relation (15) for the sequence of monic orthogonal polynomials corresponding to $w(t)$ are also tabulated to 25S for $N = 1(1)26$. Since the error coefficients k_N in the remainder term (2) may be useful when a bound on the 2Nth derivative can be estimated, we give tabulations of them to 5S.

Little difficulty is entailed in the actual construction of gauss-type quadrature formulas relative to classical weight functions where the three-term recurrence relation for the associated orthogonal polynomials are known explicitly. This is not generally true for nonclassical $w(t)$ such as ours where the sequence of related unknown orthogonal polynomials must be generated using the moments of w.

GENERAL DISCUSSION OF CONSTRUCTION

For a given nonnegative weight function $w(t)$ on $[0, 1]$, the problem of constructing the N-point gaussian quadrature

$$G_N(f) = \sum_{j=1}^{N} W_{jN} f(t_{jN}) = \int_0^1 w(t)f(t) - E_N(f) , \qquad (N = 1, 2, \ldots) \tag{10}$$

consists in determining the nodes t_{jN} and the weight coefficients W_{jN} (2N parameters in all) so that the error $E_N(f) = 0$ whenever f is a polynomial of degree $n \leqslant 2N - 1$. There are essentially only two approaches to the determination of these unknown parameters, and they shall be discussed briefly.

Before proceeding, however, we wish to emphasize in the strongest possible terms the numerical difficulties inherent in the construction of the quadrature rule (10) from the moments

$$M_k = \int_0^1 t^k w(t)dt , \qquad k = 0(1)2N - 1 \tag{11}$$

Gautschi (ref. 4) has shown that an approximate upper bound to the relative, asymptotic condition number of this problem is

$$C_N = \frac{(17 + 12\sqrt{2})^N}{64N^2} \tag{12}$$

This means that the computation is extremely ill-conditioned numerically, and that the number of significant figures lost in determining the rule (10) using the moments (11) is approximately

$$S_N = \log_{10} C_N = 1.53N - 2 \log_{10}(8N) \tag{13}$$

Thus, if one wants to have the nodes and weights of G_N accurate to k significant figures, then he must work with more than

$$k + 1.53N - 2 \log_{10}(8N)$$

significant figures. For example, to obtain a 10-point quadrature rule with nodes and weights accurate to 10 significant digits (i.e., $N = k = 10$), one must work with more than 22 significant digits at a certain stage in the calculation of the nodes and weights.

Algebraic Approach

If the nodes t_{jN} and the weight coefficients W_{jN} in equation (10) are chosen so that the formula is exact for all polynomials $f(t)$ of degree $n \leqslant 2N - 1$, then it is sufficient to choose the formula to be exact for the monomials t^k, $k = 0(1)2N - 1$, so that one has the system

$$\sum_{j=1}^{N} W_{jN}\, t_{jN}^{k} = M_k , \qquad k = 0(1)2N - 1 \tag{14}$$

of $2N$ nonlinear algebraic equations in $2N$ unknowns to solve, where M_k are given by equation (11). The drawback to this formulation of determining the quadrature rule from the moments is that any attempt at a direct solution requires the highly precise arithmetic indicated by equation (13) at every step of the calculations. For this reason, the algebraic approach is to be rejected except for very small N.

Approach Via Orthogonal Polynomials

Orthogonal polynomials play an important role in gaussian quadratures. In fact, it can easily be shown that if the nodal points t_{jN} in formula (10) are selected to be the roots of the Nth-degree polynomial $\phi_N(t)$ of the sequence of orthogonal polynomials $\{\phi_n(t)\}$ associated with the given weight function $w(t)$, then the formula is exact for any polynomial $f(t)$ of degree $\leqslant 2N - 1$; the converse of this is also true (e.g., see ref. 10).

The orthogonal system of polynomials associated with $w(t)$ in (a, b) is unique and any three consecutive polynomials satisfy the recurrence relation

$$\phi_n(t) = (t - b_n)\phi_{n-1}(t) - g_n\phi_{n-2}(t) , \qquad n = 1(1)N \tag{15}$$

with $\phi_{-1}(t) = 0$, $\phi_0(t) = 1$, and

$$\left.\begin{aligned} b_n &= \frac{\int_0^1 tw(t)\phi_{n-1}^2(t)dt}{\int_0^1 w(t)\phi_{n-1}^2(t)dt} \\ g_{n+1} &= \frac{\int_0^1 w(t)\phi_n{}^2(t)dt}{\int_0^1 w(t)\phi_{n-1}^2(t)dt} \end{aligned}\right\} \qquad (16)$$

The usual approach in the construction of gaussian rules is based on the use of relation (15) and the moments to generate the sequence of polynomials $\{\phi_k(t)\}$. After these are calculated, one solves, in some way, for the roots t_{jN} of $\phi_N(t)$, and then finds the weight coefficients W_{jN} by other means. However, any procedure that relies on the moments is essentially equivalent to solving the basic algebraic system (14) and is highly unstable numerically.

The advantage of the approach via orthogonal polynomials, as we shall see, is that the ill-conditioned nature of the problem is encountered in the determination of the three-term recurrence relation from the moments, but at no point in the determination of the nodes and weights of the quadrature rule. Since the calculation of b_n, g_n, $n = 1(1)N$, needs less than $4N^2$ arithmetic operations, the amount of multiply precise arithmetic is small and manageable, thus making this approach very tractable in comparison to the algebraic approach.

NUMERICAL METHODS

The numerical construction of our gaussian rules proceeds in three steps: (1) Determine, to sufficient accuracy, the moments M_k for $k = 0(1)2N - 1$; (2) using the moments, calculate the parameters b_k, g_k, $k = 1(1)N$, in the three-term recurrence relation (15) for the first N orthogonal polynomials associated with the weight function $w(t)$ on $(0, 1)$; (3) determine the quadrature nodes t_{jN} and the weight coefficients W_{jN} for $j = 1(1)N$. The three steps will be discussed in that order.

Determination of the Moments

The moments for the weight function of interest are

$$M_n(\alpha, \beta, \gamma) \equiv M_n = \int_0^1 t^n w(t)dt = \int_0^1 t^{n+\gamma}(1 - t^\alpha)^\beta \, dt \qquad (17)$$

for $n = 0(1)2N - 1$. If we let $z = t^\alpha$, equation (17) becomes

$$M_n = \frac{1}{\alpha} \int_0^1 z^{(n+\gamma+1)/(\alpha-1)} (1 - z)^{(1+\beta)-1} \, dz \, , \qquad (\alpha > 0; \ \beta, \ \gamma > -1)$$

or

$$M_n = \frac{1}{\alpha} B\left(n + \frac{\gamma + 1}{\alpha} \, , \, 1 + \beta\right) = \frac{\Gamma(n + \gamma + 1/\alpha)\Gamma(1 + \beta)}{\alpha\Gamma[1 + \beta + (n + \gamma + 1/\alpha)]} \tag{18}$$

where B and Γ are the beta and gamma functions, respectively. Using the well-known recurrence relation for the gamma function, we have the recursion relation

$$M_k = \frac{M_{k-\alpha}}{1 + [\alpha(1 + \beta)/(k + \gamma - \alpha + 1)]} \, , \qquad k = \alpha(1)2N - 1 \tag{19}$$

so that all the moments can be generated from $M_0, M_1, \ldots, M_{\alpha-1}$.

Calculation of Parameters in the Three-Term Recurrence Relation

There are several methods of calculating the three-term recurrence relation from the moments in the general case (refs. 1 and 11). However, for several reasons, mainly because of the volume of arithmetic involved, we have chosen a more specialized approach. We use the quotient-difference algorithm (ref. 12) to obtain the three-term recurrence relation parameters from the moments. This method will always work when the interval of integration does not have the origin as an interior point. The procedure is briefly as follows: Define

$$e_0^{(j)} = 0 \, , \qquad q_1^{(j)} = \frac{M_{j+1}}{M_j} \, , \qquad j = 0(1)2N - 2 \tag{20}$$

Generate the rest of the table

$$\begin{array}{ccccccc}
q_1^{(0)} & & & & & & \\
 & e_1^{(0)} & & & & & \\
q_1^{(1)} & & q_2^{(0)} & & & & \\
 & e_1^{(1)} & & \ddots & & & \\
q_1^{(2)} & & \ddots & & & e_{N-1}^{(0)} & \\
 & \ddots & & & & & q_N^{(0)} \\
\ddots & & \ddots & & & e_{N-1}^{(1)} & \\
 & \ddots & & & \ddots & & \\
\ddots & & q_2^{(2N-4)} & \ddots & & & \\
 & e_1^{(2N-3)} & & & & & \\
q_1^{(2N-2)} & & & & & &
\end{array}$$

using the rhombus rules

$$\left.\begin{aligned} e_k^{(n)} &= e_{k-1}^{(n+1)} + \left[q_k^{(n+1)} - q_k^{(n)}\right], \qquad n = 0(1)2(N-k)-1 \\ q_{k+1}^{(n)} &= \frac{q_k^{(n+1)} e_k^{(n+1)}}{e_k^{(n)}}, \qquad n = 0(1)2(N-k) \end{aligned}\right\} \tag{21}$$

with $k = 1(1)N - 1$. The parameters for relation (15) are then expressed by

$$\left.\begin{aligned} b_n &= q_n^{(0)} + e_{n-1}^{(0)}, \qquad n = 1(1)N \\ g_n &= q_{n-1}^{(0)} e_{n-1}^{(0)}, \qquad n = 2(1)N \end{aligned}\right\} \tag{22}$$

with $g_1 = 0$.

Calculation of the Nodes and Weight Coefficients

The three-term recurrence relation (15) for the first N orthogonal polynomials can be written (ref. 13) as the maxtrix equation

$$t\ \Phi(t) = A\ \Phi(t) + B\ \phi_N(t) \tag{23}$$

where

$$\Phi(t) = \begin{bmatrix} \phi_0(t) \\ \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{N-2}(t) \\ \phi_{N-1}(t) \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{24}$$

and

$$A = \begin{bmatrix} b_1 & 1 & 0 & 0 & & & \\ g_2 & b_2 & 1 & 0 & & & \\ 0 & g_3 & b_3 & 1 & & & \\ & & & & \ddots & & \\ & & & & & b_{N-1} & 1 \\ & & & & & g_N & b_N \end{bmatrix} \tag{25}$$

From equation (23) it is immediately seen that the roots of $\phi_N(t)$, which are the nodes of the quadrature rule, are also the eigenvalues of the matrix A. Applying a diagonal similarity transformation to the matrix (25) yields the symmetric Jacobi matrix

$$J_N = \begin{bmatrix} b_1 & \sqrt{g_2} & 0 & & & \\ \sqrt{g_2} & b_2 & \sqrt{g_3} & & & \\ 0 & \sqrt{g_3} & b_3 & & & \\ & & & \ddots & & \\ & & & & b_{N-1} & \sqrt{g_N} \\ & & & & \sqrt{g_N} & b_N \end{bmatrix} \tag{26}$$

which has the same eigenvalues as A. In reference 13, it is further shown that if $Q_{1,j}$ is the first component of the orthonormal eigenvector of J_N associated with the eigenvalue t_{jN}, then

$$W_{jN} = Q_{1,j}^2 \, M_0 \tag{27}$$

Hence, both the nodes and the weight coefficients of the quadrature rule (10) can be calculated from the eigensystem of the symmetric tridiagonal matrix (26). The version of the very fast numerically stable QR algorithm (ref. 14) given in reference 12 to accomplish this computation was used.

Calculation of the Error Coefficient

The coefficient of the error term (2) can be obtained from the g_n. From equations (16) and equation (3) it immediately follows that

$$k_N = M_0 \prod_{i=1}^{N} g_{i+1} \tag{28}$$

These constants are useful in obtaining error bounds if one has an estimate of $\max_{0<\tau<1} |f^{(2N)}(\tau)|$.

Details of the Calculations

From the formulation of the calculation of quadrature rule (1) from the three-term recurrence relation parameters (16) as the determination of the eigensystem of the symmetric tridiagonal matrix (26) with no multiple eigenvalues, it can be shown (ref. 15) that once the three-term recurrence relation is known, no error is incurred except for round-off error. Therefore, all of the ill-conditioning of the calculation of the quadrature rule from the moments must occur in the determination of the recurrence relation. With this fact in mind along with the aim of obtaining a 24-point quadrature rule with nodes and weights accurate to 25 significant figures, the following calculations were carried out:

(a) The moments M_n, n = 0(1)51, were calculated from the recurrence relation (19) in 70-digit floating-point arithmetic. The first α moments were calculated using appropriate values of the gamma function accurate to 65 significant figures. These values were obtained by the method given in reference 16.

(b) The coefficients b_n, g_n, n = 1(1)26, were obtained from the moments via the quotient-difference algorithm. This computation was also done in 70-digit floating-point arithmetic. Using Gautschi's estimate of the condition number (which is remarkably accurate) we find that b_{25}, g_{25} had

$$k = 65 - 1.53(25) + 2\ \log_{10}(200) \approx 30$$

significant figures.

(c) The eigenvalues and the first components of the orthonormal eigenvectors of the matrix J_N (eq. (26)) were calculated, using a version of the algorithm *Gaussquadrule* given in reference 12, for N = 2(2)8(4)16, 24. These computations were done using 30-digit floating-point arithmetic.

In order to check these quadrature rules, the identities

$$\sum_{j=1}^{N} W_{jN} = M_0\ , \qquad \sum_{j=1}^{N} b_j = \sum_{j=1}^{N} t_{jN} \tag{29}$$

were used. It is our experience that, for quadrature rules on a finite interval, these are sufficient to determine the computed accuracy of the nodes and weight coefficients. In no instance was the disagreement more than a relative error of 10^{-28}.

RESULTS

In tables 1 through 12, the nodes and weight coefficients of the N-point quadrature rules for N = 2(2)8(4)16,24 are given for the 12 previously enumerated weight functions. Each of the nodes and weight coefficients is rounded to 25S. In table 13 the error constants k_N, for the same values of N, are given to 5 significant figures.

Tables 14 through 25 are tabulations of M_0 and of b_n, g_n, n = 1(1)26 of the recurrence relation for the monic orthogonal polynomials associated with the enumerated weight function w(t). These tabulations are also rounded to 25 significant figures.

Appendix A is a listing of a double precision version of the algorithm used to obtain the quadrature rules. This program can be used to determine practical accuracy quadrature rules from the three-term recurrence relation parameters. In particular, it can be employed to find quadrature formulas not tabulated in this report.

The parameters associated with each table are summarized by:

Table	α	β	γ
1, 14	3	-1/2	0
2, 15	3	1/2	0
3, 16	4	-1/2	0
4, 17	4	1/2	0
5, 18	6	-1/2	0
6, 19	6	1/2	0
7, 20	8	-1/2	0
8, 21	8	1/2	0
9, 22	3	-1/2	-1/2
10, 23	3	1/2	1/2
11, 24	2	-3/4	0
12, 25	2	-2/3	0

In the tables, a ± b denotes $a \cdot 10^{\pm b}$; for instance, 0.17 - 6 means $0.17 \cdot 10^{-6}$.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif. 94035, January 19, 1970

APPENDIX A

QR ALGORITHM PROGRAM OF GOLUB AND WELSCH

```
C      THE ROUTINE IS ACCESSED BY THE FORTRAN IV STATEMENT
C
C          CALL GAUSSQ ( M, B, G, ZM, T, W )
C
C      WHERE
C
C          M IS THE NUMBER OF NODES IN THE QUADRATURE RULE TO BE CALCULATED
C
C          B,G ARE DOUBLE PRECISION ARRAYS OF LENGTH AT LEAST M WHICH
C               CONTAIN THE PARAMETERS OF THE THREE TERM RECURRENCE
C               RELATION FOR THE FIRST M MONIC ORTHOGONAL POLYNOMIALS
C
C                    P (X) = (X - B )P    (X) - G P    (X) , J= 1(1) M.
C                     J             J  J-1       J J-2
C
C          ZM IS THE ZERO MOMENT OF THE WEIGHT FUNCTION (IN DOUBLE PRECISION).
C
C      ALL OF THE ABOVE PARAMETERS ARE INPUTS. THE OUTPUTS ARE THE DOUBLE
C          PRECISION ARRAYS OF LENGHT M
C
C          T, W  WHICH CONTAIN THE NODES AND WEIGHTS OF THE DESIRED
C               M-POINT GAUSS QUADRATURE RULE.
C
C      THE CONTENTS OF B AND G ARE DESTROYED BY THE ROUTINE.
C
C
      SUBROUTINE GAUSSQ ( M, B, G, ZM, T, W )
C
      DOUBLE PRECISION B(1), G(1), T(1), W(1), ZM
C
```

```
C               INTERNAL VARIABLES
C
      DOUBLE PRECISION AA, AJ, BR, BZERO, B2, CJ, CT, DEPS, DET, EIGMAX,
     * F, LAM, LAM1, LAM2, NORM, Q, RHO, RR, ST, WJ
C
C          DEPS IS THE LARGEST POSITIVE NUMBER SUCH THAT 1+DEPS = 1
C               TO MACHINE DOUBLE PRECISION
C
      DEPS = 0.5D0 ** 54
      MN = M - 1

C          INITIALIZE W, SYMMETRIZE MATRIX, AND DETERMINE MAXIMUM ROW
C               NORM
C
      DO 100 J = 1, MN
          G(J) = DSQRT(G(J+1))
          W(J+1) = 0.0D0
  100 CONTINUE
      W(1) = 1.0D0
      NORM = DABS(B(1)) + G(1)
      DO 200 J = 2, MN
          NORM = DMAX1(G(J) + DABS(B(J)) + G(J-1), NORM )
  200 CONTINUE
      NORM = DMAX1( G(MN) + DABS(B(M)), NORM )
      DEPS = NORM*DEPS
      LAM = NORM
      LAM1 = NORM
      LAM2 = NORM
      RHO = NORM
      MN = M
      BZERO = 0.0D0
C
```

```
C          LOOK FOR CONVERGENCE OF LOWER DIAGONAL ELEMENT
C
  101 IF ( MN .EQ. 0 ) RETURN
          M1 = MN - 1
          K = M1
          I = K
      IF ( MN .EQ. 1 ) GO TO 110
      IF ( DABS (G(M1)) .GT. DEPS ) GO TO 120
  110 T(MN) = B(MN)
      W(MN) = ZM*W(MN)**2
      RHO = DMIN1(LAM1, LAM2)
      MN = M1
      GO TO 101
C
C                SMALL OFF-DIAGONAL ELEMENT MEANS MATRIX CAN BE SPLIT
C
  120 K = I
      I = I-1
      IF ( I .EQ. 0) GO TO 130
      IF ( DABS(G(I)) .GT. DEPS ) GO TO 120
C
C          FIND EIGENVALUES OF LOWER 2 X 2 MATRIX AND SELECT ACCELERATING
C               SHIFT

  130 B2 = G(M1)**2
      DET = DSQRT ((B(MN) - B(M1))**2 + 4.0D0*B2 )
      AA = B(MN) + B(M1)
      LAM2 = AA + DSIGN(DET,AA)
      LAM1 = ( B(MN)*B(M1) - B2 ) / LAM2
      EIGMAX =  DMAX1 (LAM1,LAM2)
      IF ( DABS(EIGMAX-RHO) .GT. 0.125D0*DABS(EIGMAX) ) GO TO 135
          LAM = EIGMAX
  135 RHO = EIGMAX
C
```

```
C           TRANSFORM BLOCK FROM K TO M
C
      CJ = G(K)
      IF ( K .EQ. 1 ) BZERO = B(K) - LAM
      IF ( K .GT. 1 ) G(K-1) = B(K) - LAM
C
      DO 140 J = K, M1
          JM = J - 1
          IF ( JM .GT. 0 ) BR = G(JM)
          IF ( JM .EQ. 0 ) BR = BZERO
          RR = DSQRT ( BR**2 + CJ**2 )
          ST = CJ / RR
          CT = BR / RR
          AJ = B(J)
          BR = RR
          CJ = G(J+1) * ST
          G(J+1) = -CT * G(J+1)
          F = CT * AJ + ST * G(J)
          Q = G(J) * CT  + ST * B(J+1)
          B(J) = F * CT + Q * ST
          G(J) = F * ST - Q * CT
          WJ = W(J)
          B(J+1) = AJ + B(J+1) - B(J)
          W(J) = WJ * CT + W(J+1) * ST
          W(J+1) = WJ * ST - W(J+1) * CT
C
          IF ( JM .GT. 0 ) G(JM) = BR
          IF ( JM .EQ. 0 ) BZERO = BR
  140 CONTINUE
C
      IF ( K .EQ. 1 ) BZERO = 0.0D0
      IF ( K .GT. 1 ) G(K-1) = 0.0D0
C
      GO TO 101
      END
```

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TABLE 1.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^3}$

T(I)	W(I)
N=2	
.2356502628178642264890239+0	.5652428304414819460830784+0
.8712343337313633582217530+0	.8369392748839723150919407+0
N=4	
.7337729738653181679464650-1	.1840204095652859416343037+0
.3901778818150437158234797+0	.3552780729357526979745190+0
.7105740067251409197101383+0	.4240168147000615316641873+0
.9642130006208379539527809+0	.4388668081243540899020090+0
N=6	
.3504457048024343418449511-1	.8893314050394839795691733-1
.1760429136520695590901530+0	.1883153151023423552937622+0
.3964515290272939727325090+0	.2523027875990548598314225+0
.6449294589447569740660749+0	.2829312063589259820454765+0
.8595059162165180624249422+0	.2936029070014966020373740+0
.9836370121825846374622039+0	.2960967487596860640100666+0
N=8	
.2041981505827257341810998-1	.5205969893370611545022507-1
.1046136300575156032957783+0	.1145639325324057106882327+0
.2443710169441164617315762+0	.1630713305653662725772673+0
.4210426148229107978985786+0	.1947610974432407442518830+0
.6102239468795306909477668+0	.2120368987604358993354671+0
.7846479140515148192768209+0	.2198079554326709593010943+0
.9182694706544585845538558+0	.2225846615448636726138457+0
.9906767463370753336401726+0	.2232965301127648869570040+0
N=12	
.9394755468827131857553192-2	.2403632758796658852372809-1
.4885880340937446348471456-1	.5450676200573899690045546-1
.1172862757957247918016408+0	.8170640202372461734435798-1
.2104545570059180564392585+0	.1042096714922456939030477+0
.3225648853337444699080334+0	.1213666603171629419726460+0
.4465428866931218281307671+0	.1333825536396215656186495+0
.5744120118107212077066824+0	.1410957316373476572953743+0
.6977755779244984314317850+0	.1456124880780362909417237+0
.8083877516320434331380999+0	.1479949542872908243847265+0
.8987540653327876978445911+0	.1490970902386087542590827+0
.9626935959615715229970911+0	.1495233547504520480919263+0
.9958064549069209421871208+0	.1496481092672582819391009+0

T(I)	W(I)
N=16	
.5375072026292320269517046-2	.1376992464740806390000854-1
.2810915230282973457079276-1	.3157506628720544671779414-1
.6815424584800657337585396-1	.4828438826164452830717601-1
.1240874584649518749900841+0	.6331592419950049899192826-1
.1939098213440696664295713+0	.7624917075842071955156497-1
.2751170979308180735066895+0	.8686263556627815479151271-1
.3647760092075480614302930+0	.9515522837661942764637529-1
.4596146246281561328730855+0	.1013183705051350161506304+0
.5561307640753247649933602+0	.1056708858057444537581344+0
.6507180074834021521628846+0	.1085844325240497328488026+0
.7398045472525497040930133+0	.1104228844826455937182769+0
.8199667771198717626798562+0	.1113051409012146159691636+0
.8882189486991781674602866+0	.1120091891610919185913092+0
.9418410249417674523104717+0	.1123702475631029042797172+0
.9787882928706490473955200+0	.1124858459068593553511341+0
.9976277673387387424676232+0	.1125227493785328406014949+0
N=24	
.2429278828475440309454799-2	.6229331762389439765950189-2
.1275612399629951258068546-1	.1440195242418184215649020-1
.3115782404655719027824890-1	.2235203924450986059819572-1
.5733823454055490258513748-1	.2994073872269130412501344-1
.9087096194840292993240761-1	.3705022319528801995085645-1
.1312095640297002983967439+0	.4358034988835116161605899-1
.1776963196699771916489667+0	.4945463813378257571274465-1
.2295720646487391432277570+0	.5462546105402723985799879-1
.2859872504614115078029894+0	.5907656164242609514224046-1
.3460143318228104664549461+0	.6282206888868552797110475-1
.4086615998702031901413841+0	.6590236599227909121104866-1
.4728885068696724060515173+0	.6837738739538466872604958-1
.5376223899143700972199642+0	.7031945723978392905647593-1
.6017763464176344056467814+0	.7180619634586139568817985-1
.6642678888267722075283289+0	.7291478556087100795467687-1
.7240379323996112469860652+0	.7371779959267327551023791-1
.7800696476406843922100342+0	.7428060234158851816875289-1
.8314067247073030202228529+0	.7466004105766723379620775-1
.8771705348357654398170683+0	.7490403914754268788818152-1
.9165765211377072282786839+0	.7505209773353856705695705-1
.9489474005895053985918391+0	.7513537520302204902617649-1
.9737264074306179008332503+0	.7517788655404729064110606-1
.9904868547992279873177024+0	.7519693055044354826765452-1
.9989399366660209810408788+0	.7520370465441798528666037-1

TABLE 2.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^3}$

T(I)	W(I)
N=2	
.19451791344350488443963944+0	.45671446406538605394719594+0
.72998370216048491055584364+0	.38459479912988650275781554+0
N=4	
.66200166307847495899983084-1	.16569343984106668312553314+0
.31737265578485880501497744+0	.30421337758640106866474494+0
.63733318623986218577455485+0	.27064551058128834412064414+0
.90002447585250268194033014+0	.10075693518649646079406944+0
N=6	
.32650161205075377808406284-1	.82814295516617121802977484-1
.16362978859065635898901994+0	.17360687364003363541355994+0
.36713619407333043486366744+0	.21951291523029472231767564+0
.59732997866504570342591244+0	.20127590890661327174058934+0
.80522996948502146623853734+0	.12566688061547761176584014+0
.94883288218768006588017814+0	.38432389286236193664369114-1
N=8	
.19346561679362077572729724-1	.49133318243033936798412354-1
.99018729223251721169189544-1	.10817396764031330927583454+0
.23084754046305639773301894+0	.15145045026179817178817254+0
.39692207174260233990559974+0	.17044560123996665703535904+0
.57526748775289630250067904+0	.15910038486552202776640584+0
.74316219706004950535130684+0	.11938670609201498894166694+0
.87991905913213179567296864+0	.65066588509527750939888794-1
.96906527605339582506585434+0	.18372246343095714159271574-1
N=12	
.90566475795830858224728824-2	.23169508450813790989993744-1
.47087508670329750621685464-1	.52505383503139478733973374-1
.11296864289560821661003454+0	.78495463154931912700867184-1
.20252463032774939966375214+0	.99193097590617230384381324-1
.31006805456969042809316014+0	.11266722418200385863826104+0
.42892356817120447147305534+0	.11714905179149600024162134+0
.55173192345233684073034464+0	.11156794341295059669421454+0
.67110406737586062801420864+0	.96267529577355821050262414-1
.77996124607954785099424824+0	.73507485129001766789575204-1
.87193332821233262489816704+0	.47406714962963631182754784-1
.94168909272880788336906824+0	.23224667110954660428136724-1
.98521047258201944567796234+0	.61547944820438068709699304-2

T(I)	W(I)
N=16	
.52279597885382274549992624-2	.13392715836907515034009234-1
.27336742909619454080135574-1	.30702229915468519688572814-1
.66266142453315922519221904-1	.46912621157992253245252754-1
.12060467617628654070152434+0	.61372038121696706210852294-1
.18837054985562114703215854+0	.73446446230571349542687264-1
.26709820145133149591827054+0	.82488492025992511876614794-1
.35394142434166554390826274+0	.87865012798123351005772764-1
.44579012552966387368119004+0	.89045618725824743096306734-1
.53393122773839944097143234+0	.85741222651027741631031464-1
.63146650717562311425103694+0	.78052838866333717647973214-1
.71882378644203230695010374+0	.66579157888111385902883144-1
.79846019328707515157526304+0	.52439896774075242120827054-1
.86765721335818351743207744+0	.37194073070996949809217194-1
.92406290205579232791794504+0	.22659071150849454775474544-1
.96576986858746320280372204+0	.10660299738471352562213604-1
.99136981228296052730716964+0	.27575282428262135523226574-2
N=24	
.23843175212990200512886864-2	.61139930998448219609531684-2
.12519622207255966429548344-1	.14134316948655008470450194-1
.30578206239692460186141884-1	.21933109879545202359862094-1
.56265929115466321364244194-1	.29368595373002421839877724-1
.89158752322217246336220344-1	.36310782521705915085432914-1
.12871296846730595365104564+0	.42630110078501733623395214-1
.17427491243696320609748634+0	.48194654116828375728330474-1
.22509270456593663323469484+0	.52869134616655086335494164-1
.28032992898166934466788104+0	.56516995596347501412961664-1
.33908081267999025521888174+0	.59006676582368460510124534-1
.40038637272033840115016184+0	.60222238102261211718171344-1
.46325102371686985910395994+0	.60077376543569758929878754-1
.52665923773412324580887214+0	.58530927954367011767913834-1
.58959196937953558179166794+0	.55601443861743549175421824-1
.65104265731663849473359784+0	.51378390270098532529797934-1
.71003267082178211170992114+0	.46027911020284478782102664-1
.76562686920595190841177924+0	.39791779695086727389820214-1
.81694369670718921088963034+0	.32978999237220744185375794-1
.86317606069642360574973954+0	.25950371405759915511497004-1
.90359554750356860944515204+0	.19097150900819757391497054-1
.93756712356549648913782514+0	.12815547781120446016315094-1
.96455779173056359334911444+0	.74792945390537149688937154-2
.98414452297161093996336304+0	.34127185947974781105117674-2
.99602056103061352192043994+0	.86674447363370651257610074-3

TABLE 3.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^4}$

T(I)	W(I)
N=2	
.2316591583050651934290636+0	.5516598590407890401463885+0
.8659792674539622138850256+0	.7593691181052708650860313+0
N=4	
.7277408055007991621522909-1	.1023881796117825837553400+0
.3468709465644269125272824+0	.3488285057505288890203182+0
.7061144901104944013278365+0	.3939416910848728754550179+0
.9653434133923005410740877+0	.3878704008988755570017437+0
N=6	
.3485173125557142761281620-1	.8843013075755013729046915-1
.1749682886315915633631929+0	.1866109380285485083132220+0
.3939074083530736367064769+0	.2459021033537356215535578+0
.6417379876249533992226508+0	.2659593752418354633443651+0
.8575634164124579280372166+0	.2645479871101195446906652+0
.9833630566221013906038488+0	.2595782426742706300401405+0
N=8	
.2033496172257784748583807-1	.5184036125414334056201225-1
.1041514039867219490828252+0	.1139587670679674370472625+0
.2431927128162781543395881+0	.1613008010587979044010186+0
.4190084142507212670043520+0	.1895547135053954589285090+0
.6077893865036634918859123+0	.2008061507333642056987566+0
.7828828023739938005177365+0	.2010315482594480993491039+0
.9173344734908627201617774+0	.1976915046560574000059621+0
.9905582726849314989890323+0	.1950449306108860592397948+0
N=12	
.9368466185410966624422173-2	.2396269127238288501275236-1
.4871850369540044820289467-1	.5434176851833615658554056-1
.1169317953293842353100240+0	.8138173830717595434183212-1
.2097785777560690452133465+0	.1034532045258061422468910+0
.3214931648251073087438977+0	.1195493063158403266422300+0
.4451057176953937910104021+0	.1295722726671372908621648+0
.5727818195467991104532969+0	.1343949610105444212658845+0
.6962203186863145483014786+0	.1355190713812591665308667+0
.8071635819181629481032872+0	.1345486462694183752769865+0
.8980021632416595751388542+0	.1328069309951851116629661+0
.9623889857870023748025564+0	.1312066992150097255780927+0
.9957706104214381936614374+0	.1302854866679643492262125+0

T(I)	W(I)
N=16	
.5363721599382781640284820-2	.1374075888604077635297447-1
.2804896924445798310399035-1	.3150550185729066970048261-1
.6800414276592082798779930-1	.4816555439644947104238779-1
.1238024685433812891081835+0	.6310451399806482976281922-1
.1934433319528078725628250+0	.7582072387587391102221563-1
.2744319170168170091744477+0	.8596799165278574321209731-1
.3638627314090610373066440+0	.9342061089618389715690461-1
.4585077287258034561530680+0	.9831109126770392027706444-1
.5549113307976360899691061+0	.1010050477237476364805831+0
.6494992643999194555662962+0	.1020042997633059456859069+0
.7387055431986749395530882+0	.1018591438194108022656226+0
.8191141454896716610900751+0	.1010620905693790366053031+0
.8876053678799782836168210+0	.1000121028370984633952462+0
.9414948956307087836232397+0	.9900245817370888274650027-1
.9786552959336219753890589+0	.9823110965381250431786479-1
.9976125086234840488150596+0	.9781757777520341522264673-1
N=24	
.2425835359029000382910457-2	.6220490156454103353304615-2
.1273793657151042334049450-1	.1438124932902015834845755-1
.3111288467700202621255118-1	.2231881038566384095412982-1
.5725401491664381527615139-1	.2989227748423755431295501-1
.9073412254998132762967574-1	.3697799066248598204834528-1
.1310059849837677896090252+0	.4346356351924814633463833-1
.1774118522011260053113855+0	.4925250727419372968532237-1
.2291942266157962604909415+0	.5427045317787278124923984-1
.2855077134260338168776594+0	.5847235994573115099330080-1
.3454316362080141706420780+0	.6184793122028814251605964-1
.4079833019633205772171244+0	.6442396918774921015276560-1
.4721320357156853965573961+0	.6626187866854466123890923-1
.5368142406568784689392195+0	.6745069241793548962313560-1
.6009496943254136107066142+0	.6809709585959281039026665-1
.6634590861594063194261164+0	.6831471121799231337891495-1
.7232825362240558500165380+0	.6821468633765921354974691-1
.7793986155086144429755630+0	.6789882246245587942384153-1
.8308432729883367373655738+0	.6745554813391492589377063-1
.8767280636397445933818945+0	.6695836354448859407785153-1
.9162571307971988450998696+0	.6646605534153252707928378-1
.9487424866297988725038430+0	.6602394883927204500440237-1
.9736172284448392699205196+0	.6566559244425246597530933-1
.9904464126102434692363390+0	.6541444586926673242177174-1
.9989353790030457575600607+0	.6528530366620744722669747-1

TABLE 4.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^4}$

T(I)	W(I)
N=2	
.19742379451875059621597290	.46550933019647511538180970
.73632608854899555140294480	.40850985456756482143980350
N=4	
.66718439028137876203452530-1	.16707282852707035863801400
.31649087385001437251600010	.31016492592031363363437090
.64147086733044895599394600	.28592748120417646595755360
.90185082753306379073925230	.11085394911247947857365460
N=6	
.32824491454822325649973990-1	.83267055596228050189082820-1
.16458546751521415363783110	.17504177750796209417732290
.36937952478489G0463213310	.22428744465109540336817210
.60028177216486486474357640	.21164281181764760847229130
.80741158896053578613592240	.13680974328472228295498040
.94954461084461293329239100	.42970351905377951064923620-1
N=8	
.19425201415294352940965440-1	.49516249462334041370056290-1
.99443882514066831850693630-1	.10871908375560333251463740
.23191972789840147694572910	.15294131323092000157670059
.39877184822987177477307820	.17448208647552123988968740
.57754203600658015774800060	.16677824893855675683439320
.74515603682559652480727260	.12879894597095821343962200
.88109394000752942639919410	.72066353450434144067081840-1
.96940853761256416923471530	.20716888429711885041089010-1
N=12	
.90816252432567228960687870-2	.23234139852281979810260630-1
.47220514956784807607275490-1	.52661099490156370640466470-1
.11330323471466029562485360	.78797161477852336806827536-1
.20015959107908233377597240	.99868573582860027508393200-1
.31108938660199134464253290	.11422371041713528884801750
.43027254241868198692547380	.12024144027033696467690670
.55328087599941136642817590	.11652721695287366136664800
.67262277368567221321288960	.10264181572189676405553860
.78122067139126764248114270	.80047268013222393952637477-1
.87278766647572437758909140	.52615577865255296136760880-1
.94211994686167264763717180	.26163519652491466716313700-1
.98532633817625611689198700	.69953776822508642252112250-2

T(I)	W(I)
N=16	
.52388800819167149838114270-2	.13420770824014262043946440-1
.27394593358926621495459820-1	.30768632058494851663718210-1
.66410161416082487185591370-1	.47025832836540151923776420-1
.12087737072650715544486090	.61570214076266462211326320-1
.18881557235924932695156840	.73838633087661382441764460-1
.26775004905426391843841730	.83288512868351427758290360-1
.35481026203773891267851670	.89372562356278625585837590-1
.44684586353298331405412190	.91549125965100113977853080-1
.54056240972813436465719910	.89370680744925393184151390-1
.63265247236055574122598720	.82656104636339974143709080-1
.71991691462677761592666290	.71695567188064467089122610-1
.79936954517314880258108080	.57400913619705636109526800-1
.86832625023402786416163610	.41321468482047293453418040-1
.92448093829234893346560350	.25487935102273403266624590-1
.96599555536235805663105230	.12103348408692148290965940-1
.99142189192114419555025280	.31488825092843416751616690-2
N=24	
.23876727791954797265202330-2	.61226077107959129045555010-2
.12537339155692775751343090-1	.14154477152617139741206040-1
.30821960724679749860623730-1	.21965418883467766465529490-1
.56347861898975099670736750-1	.29415540370453583958729080-1
.89291729313190966716965950-1	.36380237542195671175673820-1
.12891054068415345593305150	.42741196933893227560405330-1
.17455060612600631895846910	.48084760586951325593505650-1
.22545843820560018394890140	.53199261926935446251100900-1
.28079370609642535759608660	.57071873295317476799691790-1
.33964421920182047714207400	.59886906185270080436723510-1
.40104261439537670856478800	.61526046894983671851152630-1
.46398415621778343985523790	.61886546738209739479399410-1
.52744490954396055457515630	.60875918037888615203899220-1
.59039957690648631194668150	.58451679407386308382849830-1
.65183847876953645575623330	.54632882553908705286118350-1
.71078339586129196820861100	.49520947647977523154244320-1
.76630218941549213775190600	.43311898152470035748436790-1
.81752221973191978496548070	.36297923698791710833240690-1
.86564260231383257446728550	.28857199179387122587318530-1
.90394530876522787013211100	.21431639251357838634405880-1
.93780507237894614628998380	.14494638597819085854130704-1
.96469801904309995535004060	.85121827088247658714103170-2
.98420892226188375703344850	.39017328536416187572381920-2
.99603697727844073895010670	.99368845349804473836971030-3

TABLE 5.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^6}$

T(I)	W(I)
N=2	
.22757159795858460411106961+0	.53962507663770899671785 24+0
.85710315652426733809692 59+0	.67470024730608180 9211851+0
N=4	
.72125647724172617913447 56-1	.18071138036500867845375 96+0
.34325447109511280883530 72+0	.34033557343662105699101 57+0
.69962801602680798448888 51+0	.36450920512332906144385 61+0
.96181333801749894537157 34+0	.32876916501883200902133 94+0
N=6	
.34640640593292851677740 36-1	.87888122260099065624002 17-1
.17383713789982935135389 24+0	.18519780029135599009533 12+0
.39102632216699681709576 44+0	.24131275405634942425815 81+0
.63733570412798865834152 56+0	.25047702412295448424897 40+0
.85440979240981366742345 87+0	.23274593548970278584849 99+0
.98286267644583248818651 24+0	.21670368772332905583500 55+0
N=8	
.20241425800636588626193 42-1	.51600585945560153289140 58-1
.10365661858633879930930 46+0	.11338472765747470034655 34+0
.24194362070170472375251 48+0	.16006359457130906553644 12+0
.41665734012900298704337 83+0	.18592913215663031060172 61+0
.60452036356733315820346 94+0	.19065688174771910720315 45+0
.77966281546942305937063 42+0	.18156391127852029222489 61+0
.91576729676196730139433 6+0	.16937808942392864775389 75+0
.99035142677817238416276 02+0	.16172860116264852895416 15+0
N=12	
.93393285211345572824513 70-2	.23893938912926633449509 64-1
.48565040963975187201388 30-1	.54167477285595743904583 85-1
.11655281378333814155639 90+0	.81095677995043028446447 99-1
.20906312871336291904696 90+0	.10296015075115108877539 9+0
.32032194616897607184433 99+0	.11846489099565164317096 82+0
.44340768043695784850044 45+0	.12697018778923789885115 44+0
.57065121133309108863797 81+0	.12889478292181734024472 84+0
.69397988326915807217601 48+0	.12588240546596144897285 78+0
.80525576937242504518171 17+0	.12031552424497961836647 25+0
.89676278150812417065789 95+0	.11444821597865544187667 57+0
.96186896891214584552183 92+0	.10985151702350624518299 9+0
.99570839217295285279910 01+0	.10738019025530065523071 26+0

T(I)	W I)
N=16	
.53511138094270726084428 39-2	.13708410625288936091731 51-1
.27982584651460876682303 99-1	.31430658939463150223164 27-1
.67840912183426356345210 05-1	.48046381445235929958177 56-1
.12349792861402057031584 52+0	.62932389581178203936570 79-1
.19294919164721231053495 57+0	.75554181057753778984063 64-1
.27369609226373723928798 47+0	.85474646222382382309480 59-1
.36283982975763008977791 48+0	.92401973426307464344829 78-1
.45718475290933973752104 05+0	.96267087948388496424086 76-1
.55333891460006980232641 19+0	.97304158791979646849313 08-1
.64780522771248588274377 12+0	.96066180444433701541835 32-1
.73707310573147197764529 86+0	.93325407111992605869748 23-1
.81773019762154920706726 70+0	.89895543815157075083356 47-1
.88660274665931362637923 95+0	.84672913364963650345365 76-1
.94091247667910887351504 25+0	.83562023790972040491187 19-1
.97842729681546042476268 60+0	.81480619615896747370350 17-1
.99758611310753942730356 40+0	.80402747782396996086727 71-1
N=24	
.24220032860607720500548 37-2	.62106569950120595402754 17-2
.12717754775254975495451 32-1	.14358378248783909252635 51-1
.31063310398036634700423 52-1	.22282882004539235185823 49-1
.57161965533400636705936 80-1	.29843051442310085582847 27-1
.90586295504590769550929 01-1	.36914166159025604800100 51-1
.13078851863206228633380 79+0	.43380434726116286378246 33-1
.17710996558104358049636 77+0	.49136709086278555596687 05-1
.22879221926478058485038 09+0	.54091304469502033020620 55-1
.28496990716620646589398 77+0	.58170245681340601579395 01-1
.34478477550823046538406 92+0	.61323629966072076338860 22-1
.40720044466354048442823 40+0	.63534026834854022348387 02-1
.47121715853325859165647 32+0	.64825168551951775575171 42-1
.53578605263015147578943 38+0	.65267471487504696342949 50-1
.59984299022900444122335 25+0	.64976652663189005869110 39-1
.66232260157878480205405 19+0	.64103779767215842188088 20-1
.72217341736317365784073 28+0	.62818651177258492286268 07-1
.77837470328824356795153 98+0	.61291189537244058264667 43-1
.82995493800374797441119 74+0	.59675826276109407064700 57-1
.87601121623196625171100 63+0	.58101868506533081378292 02-1
.91572850042125572572500 08+0	.56670182786542223467872 69-1
.94839763606169560696229 84+0	.55454665392386322315911 24-1
.97343125737414791284955 90+0	.54506347732044701078354 28-1
.99037698178210396562839 18+0	.53858258429027539681131 12-1
.99892752413256248810355 66+0	.53529776022949193471375 55-1

TABLE 6.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^6}$

T(I)	W(I)
N=2	
.20059979291699934324001 13+0	.47409918865302533430402 70+0
.74557266987301237368104 23+0	.43664480430481777012845 12+0
N=4	
.67292059783793922835976 51-1	.16854470632149972279596 61+0
.31956673572757201633376 10+0	.31504041300169336657971 06+0
.64720333737522033094825 65+0	.30221748239030696644846 64+0
.90474671421008776776670 82+0	.12494138924434304860833 49+0
N=6	
.33018856434124911548150 01-1	.83765070443775958062901 45-1
.16561485927880180798264 61+0	.17629288492334648784752 53+0
.37194405130737632631044 91+0	.22787832132658545854955 29+0
.60425342528796324736445 38+0	.22189578268687402986874 43+0
.81075054870882117798183 26+0	.15115730698651781041086 06+0
.95070437283367491528355 26+0	.49754626590743359692893 59-1
N=8	
.19513068257430828647359 83-1	.49741482985053231558833 09-1
.99906592718819949899114 13-1	.10925156080996337184021 18+0
.23307511603694609664855 01+0	.15401425217902869876125 33+0
.40091989672141110180797 66+0	.17741327398554885991039 19+0
.58054918642581339074823 30+0	.17399234496280089576991 10+0
.74810941007826341658775 57+0	.14005426235550342654613 63+0
.88296425435449631057823 17+0	.81935093482434413848174 05-1
.96997536657259557582624 49+0	.24341742197510206197566 74-1
N=12	
.91095582484615589237437 78-2	.23305782897759867370832 79-1
.47367453511762335631320 86-1	.52827691680097840505219 13-1
.11366510077485270515143 41+0	.79068163278640603512189 03-1
.20383914007948821796347 48+0	.10032130208757232428089 37+0
.31219507507775108424781 83+0	.11518191057609958500494 22+0
.43186845389581906349767 39+0	.12244053323303824957338 43+0
.55328893684426207181720 42+0	.12089744874762861354199 35+0
.67477096222729356158578 70+0	.10948477840576257853925 80+0
.78313023931067883844067 37+0	.88262708542022484665449 26-1
.87414837517664319519876 78+0	.59961326026388931100735 69-1
.94282816685761189964317 21+0	.30645461848867962131988 87-1
.98552012200485731678428 06+0	.83468856339640642055917 25-2

T(I)	W(I)
N=16	
.52510919306927924945165 02-2	.13452099693997292699951 29-1
.27458860913699938569415 66-1	.30841427238578613721940 02-1
.65568022142279311841713 55-1	.47140825762205629675937 44-1
.12117135385012040500896 09+0	.61734926953795474983001 76-1
.18929123992126623065985 31+0	.74089665682871240737374 01-1
.26845611899470826895250 97+0	.83741290801880185337483 46-1
.35578795716527282754570 87+0	.90283065970575293025569 15-1
.44810794727455396351903 90+0	.93324028128104827398387 38-1
.54206474408678737296313 60+0	.92448814957388575764227 98-1
.63428284570769053658247 92+0	.87260476827089292609312 96-1
.72151371297156312610458 21+0	.77583566275153523939357 34-1
.80076476619719715033612 67+0	.63798097379865368784688 16-1
.86939263658053636347440 17+0	.47138166669601822505786 98-1
.92516488512149141869294 25+0	.29747423256635292667864 89-1
.96630215244822544462103 69+0	.14377706969048609282133 26-1
.99150952209015353540935 42+0	.37824103910520612994620 33-2
N=24	
.23914235266841034889201 66-2	.61322318642239539886928 83-2
.12557089735571725960195 54-1	.14176855382708039405686 87-1
.30670462079078683377138 67-1	.22000551671703741207073 19-1
.56437878451687664205271 34-1	.29463520858059609010632 62-1
.89436194538181057407774 29-1	.36442426919211095275373 58-1
.12912285943676322658571 56+0	.42821802288376280522503 96-1
.17484497509275216998479 84+0	.48495983944262146274686 18-1
.22584983164776311909042 19+0	.53369194502447868066462 35-1
.28129699079658890731131 44+0	.57354693100939535850840 64-1
.34027192818846751720978 69+0	.60371558870212357432910 26-1
.40180136335381089678366 68+0	.62339365318480278860932 51-1
.46487043206801758849905 09+0	.63172726336793967167487 27-1
.52844161876074391876057 98+0	.62779826473849005298298 80-1
.59147485488382996112267 88+0	.61070027038010174450969 46-1
.65294764818018022113726 56+0	.57974035893639902424970 86-1
.71187390331612023374896 38+0	.53475809262689044853924 16-1
.76732040166271607584461 76+0	.47649970092375072059747 37-1
.81842054538867827337357 82+0	.40694589301281736432678 88-1
.86438556879908720198226 12+0	.32948479549294623938185 44-1
.90451372307900784685461 50+0	.24884915456653487615158 52-1
.93819793688982973690552 41+0	.17078836691706673905959 13-1
.96493228305345196186397 40+0	.10150525006317948117954 53-1
.98431738532403565529019 10+0	.46940082787567353938499 97-2
.99606475781844137625665 87+0	.12019588538488308772986 54-2

TABLE 7.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{1 - T^8}$

T(I)	W(I)
N=2	
.22539606718959225549652314+0	.53391677868811496596554974+0
.84991618353275899863347964+0	.62967579253015440933696844+0
N=4	
.71762952639000869915295864-1	.17978811313032777055616749+0
.34133231158996667922573164+0	.33778409782399450651551744+0
.69525623993019721772025040+0	.35129125634393168552754574+0
.96046910205241626481984904+0	.29472910392001541270328014+0
N=6	
.34521394773905346757859934-1	.87583590924913555232466474-1
.17321777365984753988297864+0	.18449834652011713750869674+0
.38947471776300291382810974+0	.23974941848616405641149864+0
.63454124207168189551364444+0	.24425276289401140224705054+0
.85192949843591461026016334+0	.21598923078505138212118124+0
.98246066766474904742117544+0	.19151922160801184178162484+0
N=8	
.20188386826810265855866134-1	.51464688357330522331224364-1
.10338030728069329650409064+0	.11307495457702752986833704+0
.24126952265340323449533304+0	.15957631268537630151016764+0
.41537744549386626827531844+0	.18480370882951339689265144+0
.60250359811577138053081424+0	.18694766917542079817001224+0
.77745633965161345664539044+0	.17215640723689182043842874+0
.91447187235169044136355634+0	.15353958612100969770050784+0
.99017044169643517783279574+0	.14202924423569930838464714+0
N=12	
.93227537457263756758247574-2	.23851466511637861162921174-1
.48478239621508044599244334-1	.54069760193367496363717054-1
.11634139736098625244817944+0	.80944248747037649509637754-1
.20867315859578831040838764+0	.10274963856187300943680704+0
.31969318525928044761057854+0	.11813410554363295741506824+0
.44247029324719162661809624+0	.12623993197428006311091624+0
.56937168125787215989834504+0	.12703438590613499380116294+0
.69247351804090742405650644+0	.12180867741161016433102834+0
.80383056633833171945310574+0	.11321859900266755434110804+0
.89576093975378497544970264+0	.10432626122473060375829794+0
.96142729025266636921306194+0	.97440063425404979163435164-1
.99565428961534388137008874+0	.93775432715892042908418654-1

T(I)	W(I)
N=16	
.53439320294384032616913414-2	.13689996099537138946103404-1
.27944882672469469478292904-1	.31388100480711852064308144-1
.67748812863300229407032974-1	.47980253447969253975905634-1
.12332814908558312765237784+0	.62843074308524713511999314-1
.19267848184047408756792124+0	.75439362100465092864692724-1
.27329937453055954067721074+0	.85318187274877390507000884-1
.36228803747701789915945424+0	.92139487888937684278203344-1
.45644839112279211862088304+0	.95721018371560371638119014-1
.55240649809901684179942454+0	.96127135497132576042483594-1
.64671434736512797142805404+0	.93777237248127886153227394-1
.73592926852806447844351204+0	.89471419365316750073827244-1
.81668600605147288329618484+0	.84238726751604396322334714-1
.88580108014063817766396324+0	.79070489072062365397668734-1
.94042712154344368943378884+0	.74715159820314580185209824-1
.97823226215451055822803944+0	.71629742931510428329926814-1
.99756324813642435349002244+0	.70043180559616895019608414-1
N=24	
.24198177105678068452546964-2	.62050503758064001671511484-2
.12706258824243254565573344-1	.14345371457262162838794114-1
.31035141991776176812737834-1	.22262563979695071274673754-1
.57109878826201680211603334-1	.29815556138274848207098544-1
.90503182031535696749792544-1	.36879615545539267781817274-1
.13066736943919770623509664+0	.43338793028205681301617404-1
.17694373221097005227883104+0	.49087289687354150633028644-1
.22857353248444360767231544+0	.54031285857918318422283444-1
.28471070024201954086536954+0	.58091053720073715244650464-1
.34443608733020322159191024+0	.61203979451154971465691454-1
.40677296340389552060335724+0	.63329667747362902248672494-1
.47070332044154116100176334+0	.64458308127513393876754324-1
.53518365014025360918940474+0	.64621994331087631197919514-1
.59915987288385506471895284+0	.63905711166230332341261724-1
.66158021476110357577573714+0	.62451442656491247109596644-1
.72140709721282499410418014+0	.60449710769563462166278754-1
.77762962964428028169902754+0	.58117966321928283508834944-1
.82927854864131615463465364+0	.55672585917290555522110064-1
.87544314692433201345142044+0	.53304242845877175748303424-1
.91529753189371996059847404+0	.51163632758369061630984354-1
.94810985321337197580241504+0	.49358666026646625039029054-1
.97327342508478379801846804+0	.47959634061339976125623494-1
.99031741126444288811272374+0	.47009164052035983394332574-1
.99892074872264005433590584+0	.46529088587216763192654164-1

TABLE 8.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{1 - T^8}$

T(I)	W(I)
N=2	
.2023750685362043002080766+0	.4785543938403536799790999+0
.7520391923118905767124972+0	.4523196631342618202629146+0
N=4	
.6762020666634779647513202-1	.1693763823459224936299348+0
.3212534163869693801322080+0	.3171120744648610639090086+0
.6509310676023403155218855+0	.3099764357293958421252360+0
.9070043041796462032666809+0	.1344091644344361005778351+0
N=6	
.3313027475668341321183687-1	.8404924191262153790006751-1
.1661895518552647647412989+0	.1769342865841215075658857+0
.3733520725141671633334912+0	.2291773225223383328082682+0
.6067400623248180999823801+0	.2262440790501443410582448+0
.8132212908592681773150881+0	.1597483024602636474666288+0
.9516393975671376047068676+0	.5472082445126133442919461-1
N=8	
.1956344262078435102748308-1	.4987029123993482666537866-1
.1001683179778967273816619+0	.1095438518452178839518468+0
.2337093202421058867791785+0	.1544817045074620830549787+0
.4021048109961527789866053+0	.1783663980062306759258848+0
.5823926129399935655955182O+0	.1768156337802232964489838+0
.7501944930315226764644000+0	.1461019285986481288694083+0
.8844225594638400888269185+0	.8858634114313400008030479-1
.9704398364684984369409855+0	.2710790785376460524522864-1
N=12	
.9125565122275229627343619-2	.2334679306084943762837985-1
.4745121654476581940468389-1	.5292189356715093855202448-1
.1136687866611254531722562+0	.7921359243826427639806870-1
.2042136779918064104489568+0	.1005217345566718070102287+0
.3127954783581012200101503+0	.1154864769856860636827039+0
.4327561644352037276574994+0	.1230779583387508235723436+0
.5564929829573438845021162+0	.1224393795865288369901904+0
.6761952480091170977483304+0	.1126179475025951044298217+0
.7845189000322064234013185+0	.9295187663587300418967146-1
.8752077517796296135575088+0	.6488855002336285321680844-1
.9434040098971910195901522+0	.3399186635382039889602390-1
.9856813991375511742739003+0	.9415987925061955675749333-2

T(I)	W(I)
N=16	
.5258068642331205592011S3-2	.1347003381579255481527408-1
.2749557016350563490927281-1	.3088284802541602794364957-1
.6665763971437459200311490-1	.4720509826544924450457275-1
.1213363831011425383909180+0	.6182151044238530190608770-1
.1895539167678606151586221+0	.7420031616977847969888565-1
.2688399770836495414400037+0	.8388976842622197297529103-1
.3563196451289120420621755+0	.9052494278522999726970196-1
.4488140095124140749371113+0	.9381115409519722873095513-1
.5429554542279610308100100+0	.9346690447570131313307361-1
.6353254032171594767530392+0	.8915656592941315824227053-1
.7226164732498821281489782+0	.8053857320649268033350802-1
.8017948431570569432517410+0	.6757155875016629732728073-1
.8702219156123662880283397+0	.5102490594583470374182135-1
.9257169023530737913066463+0	.3286609860273077795702424-1
.9665772142195598758646960+0	.1614968672176494128266399-1
.9915829899015769048094784+0	.4294096317040820380014180-2
N=24	
.2393570728835934495465314-2	.6137739908750204505250539-2
.1256832668543978019431O9-1	.1418963084148020463342254-1
.3069812789406790296946786-1	.2220050085720592078733463-1
.5648902164668579476112148-1	.2949060120632673188358820-1
.8951777034604105274728961-1	.3647629974569800405851335-1
.1292417016676000635154758+0	.4286256673848763307263197-1
.1750079172634822436804181+0	.4854422827242185549232512-1
.2260639583677459347171650+0	.5342745042256629671156613-1
.2815699755360449733519266+0	.5743072940241682907496560-1
.3406122049456964283938684+0	.6048467770287235681060999-1
.4022176008435241666455611+0	.6252950454379717638162151-1
.4653696792025484559462736+0	.6350917747744361379165615-1
.5290259041485420875628999+0	.6336280525153360015031298-1
.5921370084927656173952666+0	.6201725056325887133803173-1
.6536681306503592327856844+0	.5938894013913863303715867-1
.7126205920123813369673510+0	.5540377482982729789077872-1
.7680522014950836521114354+0	.5003878750630705283204224-1
.8190940033883210571707650+0	.4337856431691832145559328-1
.8649624961387718617281988+0	.3566890077889422388875182-1
.9049677505743729235272227+0	.2734596744591389112474242-1
.9385186698060503050547735+0	.1902394494156246980952381-1
.9651266397406279838707362+0	.1143554180223441764093889-1
.9844086668641725384371393+0	.5333525376598788243009912-2
.9960882103028663319616230+0	.1372946902961105627644295-2

TABLE 9.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/\sqrt{T(1 - T^3)}$

T(I)	W(I)
N=2	
.13030212209544721459062D3+0	.13921292835230428515247D3+1
.83529407216138547308672D9+0	.10365213643645387602952D8+1
N=4	
.35671255929321028328766D8-1	.74718803638573653585553D3+0
.29395607105705407343223D6+0	.65867939467457505083332D3+0
.67630178065008426848115D4+0	.54512469340042641886D9391+0
.95937324527976128376828D2+0	.47765852342684360627D3239+0
N=6	
.16301374885703830402376D9-1	.50807622319474744247831D2+0
.14078971261482940743369D0+0	.47773256468045284073853D6+0
.35974690189268913324429D5+0	.42628176024192021445945D0+0
.81850779705526313863274D8+0	.37255314352827577073298D0+0
.84776277232455293406504D5+0	.33229973556970488494079D2+0
.98219387082043990520160D6+0	.31172522067248045846984D97+0
N=8	
.92919416265619055199630D9-2	.38443163102999284360737D6+0
.81860481710915338570209D8-1	.37091157361535335626499D3+0
.21624714877382333737230D57+0	.34584232185337699273176D27+0
.39403896631792709628330D99+0	.31437030921237415437381D79+0
.58912687847921857600527D8+0	.28328559909206946220586D3+0
.77178565614465075836245D36+0	.25779586552178961381626D0+0
.91309351543168304695808D74+0	.24086709225247625171604D4+0
.99006784086624089944335D59+0	.23164625531014893710246D15+0
N=12	
.41827976546135112628281D3-2	.25834716320836585733143D30+0
.37238635097742769588145D49-1	.25416022903119879102050D67+0
.10121562916695302717485D44+0	.24596379282999366203553D72+0
.19199964513046286506080D37+0	.23441010574684727280181D43+0
.30368765781553923057575D87+0	.22054481392031998757800D12+0
.42899176217290142145688D07+0	.20582789481270250266878D37+0
.55953110706833536761194D61+0	.19165743912462766737711D97+0
.68639678330957108632949D38+0	.17908970879310247038175D12+0
.80076530321461662330360D92+0	.16875695063365951714413D16+0
.89456572755377073159629D59+0	.16095027048134834153659D15+0
.96111102184681153772960D35+0	.15575508634092377654983D44+0
.99562359519731607765005D2+0	.15316719296349176539443D73+0

T(I)	W(I)
N=16	
.23674072325736759232548D78-2	.19447357198268039510392D45+0
.21175263412055072217662D48-1	.19267524641495637686119D77+0
.58096294130847535125231D75-1	.18911047441821359180248D64+0
.11176906212560046292547D49+0	.18389798772173254043725D31+0
.18021564805455819765113D65+0	.17726775473082973078615D96+0
.26090753467450813370673D23+0	.16958403133728046388824D97+0
.35084437140499452965218D78+0	.16130652501403090264703D97+0
.44664814205073678795547D84+0	.15291862885459709852364D69+0
.54467675862045516881848D04+0	.14485321553056280982682D06+0
.64115764197083409867533213+0	.13744740780795479234126D17+0
.73233646105131249641122D5+0	.13033019257384148088644D38+0
.81463237003881999250679D4+0	.12543595999568097013884D11+0
.88478996676355981841536D2+0	.12102946547715583490618D20+0
.94001901568222577118002D49+0	.11773169313367636566007D26+0
.97811458263369632596479D45+0	.11554117818127485484028D22+0
.99755190867427174419956D2+0	.11444920471305340307008D01+0
N=24	
.10585512565991761888735D21-2	.13009624891442717677900D83+0
.93004706365770494383433D33-2	.12955336244216905711986D63+0
.26243566863235878464058D58-1	.12847144076242735935329D48+0
.51008904680449521299951D45-1	.12686097424793300842300D80+0
.83384256374132115525961D18-1	.12474366519168118223251D91+0
.12283102243560641927502D15+0	.12215708017409835953129D89+0
.16869282565842717146691D56+0	.11915752735092823763589D8+0
.22020552972338801304059D74+0	.11581961189868420475906D16+0
.27650859684590514170583D45+0	.11223353364223643819196D16+0
.33665784708949005285536D94+0	.10849670320671348241322D17+0
.39963974222799696619287D56+0	.10470820745803951506795D62+0
.46438726301402580253491D7+0	.10090078921316526467022D78+0
.52979707961246230446152D2+0	.97335779887771871858492D27-1
.59474957154549981368295D00+0	.93900128833731420267382D19-1
.65812511679903897539501D10+0	.90705628982329749933119D7-1
.71882836746693525771852D19+0	.87789834797683950170640D1-1
.77580501512069481940150D29+0	.85178033458760723469316D14-1
.82806234944879539411007D8+0	.82885688705560515252260D3-1
.87468735955514662666672D96+0	.80820927737256856263995D68-1
.91486360522787700004769D0+0	.79286805960301284077035D61-1
.94786311177436931585868D2+0	.77983219632005723706473D45-1
.97317539021819268324642D17+0	.77008428026922098377748D3-1
.99028615232749996972719D7+0	.76360200696130135421195D7-1
.99891750587947611175518D2+0	.76036626643913459477911D7-1

TABLE 10.- GAUSSIAN QUADRATURE RULE FOR $W(T) = \sqrt{T(1 - T^3)}$

T(I)	W(I)
N=2	
.26816581655995979649512 41+0	.24339216373783495583748 54+0
.76653615516379321955405 09+0	.28020661186046391723962 18+0
N=4	
.10043631271162057413955 53+0	.61224037194592825174449 98-1
.35836746941039164157773 29+0	.17758002149781938574354 79+0
.66666204441274906806532 13+0	.20074560564251773684896 41+0
.90901169958926974492447 75+0	.84049111263368925310145 18-1
N=6	
.51410049637323424712255 38-1	.22881267399823755886342 77-1
.19447904578578793151744 73+0	.79055374403344082536567 66-1
.39835517648889100770049 24+0	.13474336885321657888865 73+0
.62063126642700869482018 25+0	.14882322967944621604703 43+0
.81758288349056435555365 50+0	.10417754191795257252708 27+0
.95223914864556712495522 86+0	.33917993344515667191422 44-1
N=8	
.31060743307199164696615 29-1	.10828660021945029782891 93-1
.12022010895639767578715 86+0	.39811534784624137480681 79-1
.25592194504092790563246 33+0	.77019253417570879550760 93-1
.42078351671328830425713 75+0	.10778445257158239155943 64+0
.59429718711959569988333 99+0	.11703953987967671667694 44+0
.75564871462903975229427 77+0	.97492368456761363502774 3-1
.88606894677171482932652 22+0	.56911877092483589009551 09-1
.97069640791075794196360 31+0	.16711089373654765496563 22-1
N=12	
.14833519539001095612875 33-1	.35947648724582274435000 04-2
.58429169060280014955968 42-1	.13832113021318334823972 21-1
.12812243303408577621912 78+0	.29120997197411499453269 90-1
.21965312519032710533027 47+0	.46937470450036006113967 56-1
.32745804674184969731911 81+0	.63989408056344838742312 3-1
.44506815278834643268067 27+0	.76584625911246717296467 34-1
.56554934587160722120495 66+0	.81384364853788387279781 76-1
.68192278909617637699205 25+0	.76473775375907000316084 39-1
.78756019110361543714194 53+0	.62337013550878488446928 70-1
.87651537416625793402253 92+0	.42195465669738484626570 63-1
.94382624890213444792695 77+0	.21372839301012518257044 53-1
.98576017495526866108752 11+0	.57759373381583702932888 29-2

T(I)	W(I)
N=16	
.86500147622604323567056 27-2	.16042452080955559277911 43-2
.34294480982810004865114 02-1	.62755728365515901136275 59-2
.76026376325800039082551 238-1	.13598233448459880707048 96-1
.13236783243334789782903 52+0	.22901915256573475190985 17-1
.20132438782629440039710 184+0	.33284213542364351700879 19-1
.28046335168074173945003 25+0	.43640461827882571535772 64-1
.36701186317970446684949 02+0	.52724845963910974293784 56-1
.45796629344661103948456 91+0	.59271690314543834412157 07-1
.55020429555667067060314 93+0	.62185563175360356380973 18-1
.64059349453074176697889 32+0	.60772269556195453846696 95-1
.72609396420079005794565 73+0	.54950575307498465697760 28-1
.80385343507228328965786 98+0	.45737527512354500698905 5-1
.87129449183491064724484 07+0	.33405897236707276921674 15-1
.92619260570597612727085 01+0	.20934505566925854589847 17-1
.96674339153807342109994 56+0	.10045813916731822253682 74-1
.99161729137193384410670 73+0	.26292216896195943453590 7-2
N=24	
.39856441620131976700994 19-2	.50256411242608908807077 82-3
.15878147662431565905336 62-1	.19899690365580262411026 45-2
.35485174598117109620225 26-1	.44019850100855396112037 84-2
.62489384546261633806343 80-1	.76400918976467234296069 40-2
.96453407586453340914734 29-1	.11569572544715210796844 64-1
.13682701428511747198553 17+0	.16021536923035650146032 56-1
.18295651340661135404789 40+0	.20794880142558768280405 8-1
.23409624748701596469264 58+0	.25658945814517502049210 94-1
.28942187324482683938565 94+0	.30358332002384294312321 19-1
.34804497492048409191061 92+0	.34621480068157562143264 63-1
.40902850837432109173986 92+0	.38174238122168670709919 85-1
.47140261496814933088833 71+0	.40758559237039945453381 20-1
.53418044301714623108774 73+0	.42155164659654232846156 35-1
.59637372430228840647012 09+0	.42207745063035557973263 69-1
.65700793851048891028853 30+0	.40845411215442220661902 39-1
.71513694595811371777509 42+0	.38099832799253506183528 95-1
.76985698306197147867842 01+0	.34113867632450657072769 07-1
.82031990868417008926659 20+0	.29139408104744752282789 18-1
.86574557578851475359954 62+0	.23523498202667697892198 08-1
.90543319100185080225398 53+0	.17683297335043979745125 28-1
.93877151940458882972148 03+0	.12071965732676861974961 71-1
.96524779457470520880010 33+0	.71388124008803354648282 75-2
.98445520406942196353416 57+0	.32879084219191932791233 96-2
.99609883683388681640630 64+0	.83970911923589543087211 59-3

TABLE 11.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/(1 - T^2)^{3/4}$

T(I)	W(I)
N=2	
.26511046973516099905292214+0	.66919130283172978494969684+0
.93328991578435802970643404+0	.19528662514603900025517871+1
N=4	
.77578482444002621920837271-1	.19582611298213688147112294+0
.37092256668574864615304394+0	.40953253380491261879589794+0
.74342964922391992267456014+0	.62273169363369108031392334+0
.96276767632287712397424064+0	.13939672138713792298838964+1
N=6	
.36336435145556056187995808-1	.92379362974406466305823914-1
.18292249396611746184752394+0	.20042575657461027849663774+0
.41161357596796484383131374+0	.29004692106483496685159004+0
.66586195468113116572480894+0	.37965693043938415756850324+0
.87879624913350164572360704+0	.51945529864223372675466614+0
.99228650339093897686396164+0	.11400932645966502144876194+1
N=8	
.20975411123145326545331284-1	.53514924333774495486340054-1
.10759706825597412795812454+0	.11893329689581529359060204+0
.25151791038796482000091704+0	.17496504497011846112937134+0
.43288729033058504321243724+0	.22397184639192541862457914+0
.62540162547447986841728534+0	.27338547626164583629640134+0
.80016032481978710332125784+0	.33696787826051012588809844+0
.93035551575201293867189934+0	.45231024160773871106940634+0
.99564873139504534519992954+0	.98798884557059146837804114+0
N=12	
.95627935633959770121247044-2	.24471065085487920438431624-1
.49756723910748981761377564-1	.55637912043116580506654984-1
.11951151106849944795956574+0	.84156590865996126216177404-1
.21452138781998124566462964+0	.10955743650394456232761224+0
.32874473450640321522208454+0	.13232107249655039087125994+0
.45473648567644849664133934+0	.15363697223114951502774394+0
.58411885623567274157127454+0	.17526782868049812784206374+0
.70815853403112656747975464+0	.19972728414843376147119944+0
.81839234068001993009194814+0	.23116494727920237531519384+0
.90724248605534572925700804+0	.27849064818180266963937614+0
.96856765994287935969096554+0	.37050408362586991696738614+0
.99806145669488076135403794+0	.80712171315006786359672054+0

T(I)	W(I)
N=16	
.54467440441263105376372194-2	.13954627690976410838955154-1
.28490310765834251110361744-1	.32030632693471503308205964-1
.69100807354588366408833774-1	.49146878322904436280189754-1
.12585092609329211755378944+0	.64962500295090999622078754-1
.19670524226075528145619184+0	.79403285757797902470406354-1
.27906207286684014660766124+0	.92621525359134537328018984-1
.36993586391797630718646894+0	.10494958326295434670679354+0
.46586488698443568168896264+0	.11685975443437986916194614+0
.56323971723542339288121754+0	.12895532231441259618562694+0
.65834555143555595058514564+0	.14201266944955869796441154+0
.74753075731768617555595364+0	.15711420094716846597936394+0
.82735423970520748500644804+0	.17597319080610628456878654+0
.89472477511806154247369454+0	.20178330740569979404794784+0
.94702611474774873548991744+0	.24194957181956576669705794+0
.98222160619717713221427674+0	.32119156290100801257327624+0
.99890844757901095588296804+0	.69914899576789018673177434+0
N=24	
.24507607081204803967603274-2	.62845529467557349706650034-2
.12869634814090201104864974-1	.14533356815702051108163074-1
.31439216727617078332072074-1	.22574489148050907849260274-1
.57864384108950716979978074-1	.30297460737518360344221754-1
.91718538843130913204754734-1	.37632729847635755626881544-1
.13245089954272573089103904+0	.44545693452492064184040394-1
.17939410444655866694255144+0	.51035689355848627896993524-1
.23177361471929502590946044+0	.57132585615389521191306664-1
.28871934956181436479564924+0	.62892044106966783813224444-1
.34927945878956781323228954+0	.68390850561211083865534224-1
.41243596948132351410859264+0	.73723448684719269486171814-1
.47712211113443152877638964+0	.79000528277782774294870194-1
.54224051159154169458274004+0	.84350392748174670711519544-1
.60668255341357906990745084+0	.89924005150185797914774384-1
.66934777776262744817472694+0	.95905256632536434027069344-1
.72916335264526841759926874+0	.10252948592105769541800144+0
.78510307952457402879736474+0	.11011652648135262795244304+0
.83620559397711726533505114+0	.11913204059858648996058164+0
.88159142073096034353404164+0	.13030989538560433433533624+0
.92047856967456081438157274+0	.14492270034936439738142014+0
.95219638252517476846669294+0	.16547639614186421745834244+0
.97619732955354143628283864+0	.19793851005533217979146254+0
.99206609665377920880254314+0	.26243757251662646489131864+0
.99951442174846767541872944+0	.57097137876356038310777564+0

TABLE 12.- GAUSSIAN QUADRATURE RULE FOR $W(T) = 1/(1 - T^2)^{2/3}$

T(I)	W(I)
N=2	
.25709789481903027720432l4+0	.64198491158096539751936954+0
.91354120859392987561064194+0	.14612882464072159942431594+1
N=4	
.76532339381824329463810754-1	.19298565869966032428130B+0
.36561543277571101819602104+0	.39749896350254752725585334+0
.73401695919370996488721314+0	.56802565109219989925603964+0
.97692040262217990787555144+0	.94476288652346793282250504+0
N=6	
.36021960177060514417817184-1	.91555501567090311098462264-1
.18129663463205099028492054+0	.19788125196072191921164964+0
.40760019157265115172003134+0	.28231921909696969659268558+0
.66015005995623856663533074+0	.35649737385481328650824754+0
.87302262496105616196661004+0	.44957450950988183658691594+0
.98956296269533948719957624+0	.72544530197697707243039754+0
N=8	
.20841649086656504656541614-1	.53168209470726822966042774-1
.10686720337884462618512354+0	.11799435839850302423248274+0
.24960320885291654082524794+0	.17268341278522137625641974+0
.42995057440138407061863094+0	.21824908385864885820941524+0
.62136521956016697138995304+0	.25973101615484583305856634+0
.79572273459400771097052124+0	.30567488397143240631281314+0
.92665206868729290273934l+0	.37531662984038249341355174+0
.99406316874676917400277904+0	.60045556350842057733595514+0
N=12	
.95227986064090474381315684-2	.24368059649414548295326644-1
.49544931278944221141367264-1	.55384093043102841275129684-1
.11899016978827631725532114+0	.83668899482336790917264614-1
.21356453227953416381580664+0	.10859258127461556761849914+0
.32726255322376723275372504+0	.13037682639668180415472604+0
.45270486796099097673639084+0	.14984564963845641551171334+0
.58160471969246064633975964+0	.16823094889591178239457664+0
.70533280124561806948726364+0	.18716778784855806254759754+0
.81552671198490535442256094+0	.20906791181631161692615294+0
.90468910755073252921312184+0	.23852733416465202107999334+0
.96672479485257230159620744+0	.28868463376713867747805274+0
.99735123651280820619506504+0	.45935843201100126356349624+0

T(I)	W(I)
N=16	
.54297884134169683807328024-2	.13911038852786877168243644-1
.28400687156745907959234104-1	.31926353972153045586428384-1
.68879852723204083851624624-1	.48965172698252989845769834-1
.12544095563172805853825834+0	.64652664936025854254433674-1
.19605361431685522201715954+0	.78856713856052419470040004-1
.27814734175003043133838024+0	.91648378420425152964308304-1
.36869493925335920229384164+0	.10325538771234359370317844+0
.46431984222144305265707664+0	.11401658062791003700848404+0
.56142301782146423826721344+0	.12435570707422388270016784+0
.65632281048638259613203664+0	.13478768127584157794363064+0
.74540052040528674438988084+0	.14597458142460576472929054+0
.82524466714888227901461004+0	.15887619531930461616517804+0
.89278740425414017678068574+0	.17513914154939013016242274+0
.94542717369980617539286604+0	.19828980876299243919712074+0
.98113193512192133898870624+0	.23899291893526099855403114+0
.99850008598920463603617104+0	.37962508257061207230980114+0
N=24	
.24457109351078227974539484-2	.62715853509166223047674174-2
.12843187509272837940152424-1	.14502881278739267496394824-1
.31373585284907023743084804-1	.22524814113788505882739534-1
.57742290621261895444048874-1	.30223381827123322772080264-1
.91522659908316019774769784-1	.37522586344618993546615724-1
.13216453652415369108659754+0	.44378127602942666401013324-1
.17900187108911446799210884+0	.50776433278717367293387764-1
.23126229263188854816750804+0	.56731425173743454064681854-1
.28807883786848333700847694+0	.62279866541517811924983774-1
.34850373448555100347550544+0	.67476613702925654308896364-1
.41152399713841017717177054+0	.72390501825254977267494634-1
.47607851080212074168359264+0	.77100817636572541691103654-1
.54107622831494574517553004+0	.81696310349461142853420654-1
.60541508785354644229087344+0	.86284593806432926536823914-1
.66800125311071673676827014+0	.90977472807939324753225634-1
.72776828740687132295919894+0	.95919357755479276055444754-1
.78369588843387451480282144+0	.10129295375340910441384574+0
.83482783030680943936388294+0	.10735016071433306332771654+0
.88028878280530968240681614+0	.11446849644955378174734574+0
.91929970311495941626536814+0	.12327226996654709926541144+0
.95119151999406489743448614+0	.13493175182405643114975634+0
.97541683529472888693993604+0	.15207407830409702583061634+0
.99155916357687252100491864+0	.18280161565716825491627694+0
.99933339981432698624851474+0	.29002322234865440981262344+0

TABLE 13.- ERROR CONSTANTS k_N

Weight function	N						
	2	4	6	8	12	16	24
1	0.88323-2	0.35333-4	0.13925-6	0.54651-9	0.83793-14	0.12817-18	0.29917-28
2	.38568-2	.15381-4	.60574-7	.23766-9	.36431-14	.55722-19	.13006-28
3	.83723-2	.33428-4	.13195-6	.51784-9	.79396-14	.12145-18	.28347-28
4	.40962-2	.16231-4	.63926-7	.25081-9	.38488-14	.58808-19	.13726-28
5	.78509-2	.31380-4	.12366-6	.48528-9	.74402-14	.11381-18	.26564-28
6	.43400-2	.17319-4	.68214-7	.26764-9	.41029-14	.62755-19	.14647-28
7	.75454-2	.30175-4	.11891-6	.46663-9	.71543-14	.10943-18	.25543-28
8	.45125-2	.18010-4	.70939-7	.27834-9	.42669-14	.65265-19	.15233-28
9	.18611-1	.72707-4	.28401-6	.11094-8	.16925-13	.25831-18	.60142-28
10	.20227-2	.79014-5	.30865-7	.12057-9	.18397-14	.28071-19	.65359-29
11	.13386-1	.51613-4	.20089-6	.78342-9	.11935-13	.18198-18	.42341-28
12	.11944-1	.46749-4	.18286-6	.71488-9	.10918-13	.16668-18	.38828-28

TABLE 14.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 1

[Zero moment = 0.1402182105325454261175019+1]

J	B(J)	G(J)
1	0.6150198678198335788169563+0	0.[illegible]+0
2	0.4918647287293942058938206+0	0.9719998072864766978734869-1
3	0.4942420946542773527867192+0	0.6480442960076583929694835-1
4	0.4972154973440492686035494+0	0.6347162017942392756940588-1
5	0.4983553817384675085928096+0	0.6302665582117424823634536-1
6	0.4989138302174447250865230+0	0.6283096286916349924533615-1
7	0.4992291580192525787958213+0	0.6272727656339084492637858-1
8	0.4994245960826758460064597+0	0.6266571647154172186701221-1
9	0.4995540782953546455435342+0	0.6262618415588194270540389-1
10	0.4996442788137826402990677+0	0.6259929023366112311948653-1
11	0.4997096265296906928777585+0	0.6258016629848901163648707-1
12	0.4997584830310309334443183+0	0.6256608214529109093540000-1
13	0.4997959684415942918933535+0	0.6255540992892579903320774-1
14	0.4998253524094227033312671+0	0.6254712991341269109622364-1
15	0.4998488165401011149931981+0	0.6254057684446898891785706-1
16	0.4998678497832047459205261+0	0.6253530168863810152861730-1
17	0.4998835016646389778878341+0	0.6253099244959812457538696-1
18	0.4998965283447508966728075+0	0.6252742686957877446515471-1
19	0.4999074858084827709476154+0	0.6252444314072017553104359-1
20	0.4999167902564709870214112+0	0.6252192119340475533176225-1
21	0.4999247582582129859329286+0	0.6251977041042140630985484-1
22	0.4999316340639115500779560+0	0.6251792136294655017448687-1
23	0.4999376085632916995228546+0	0.6251632013092715978563307-1
24	0.4999428326887550657745987+0	0.6251492432368882826861675-1
25	0.4999474270486818254884930+0	0.6251370024294201042101103-1
26	0.4999514889558611242126238+0	0.6251262082814268729665467-1

TABLE 15.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 2

[Zero moment = 0.8413092631952725567050114+0]

J	B(J)	G(J)
1	0.4392999055855954134406831+0	0.[illegible]+0
2	0.4852017100183943815548000+0	0.7115415880902764271108506-1
3	0.4952340721964686178118784+0	0.6442813666042976091605580-1
4	0.4975594061782398313079980+0	0.6332966745806871531576270-1
5	0.4985136922886463805089016+0	0.6296973592602236653531604-1
6	0.4990001839394647825814603+0	0.6280211458494534797182123-1
7	0.4992813856085603058674026+0	0.6271063547685535342098320-1
8	0.4994585632933782518982225+0	0.6265525449907410750921175-1
9	0.4995773992440699269037173+0	0.6261918246049314627814951-1
10	0.4996609771701953955817794+0	0.6259437591540021127428069-1
11	0.4997219907331511660646697+0	0.6257658535884642161731121-1
12	0.4997678923911061752064208+0	0.6256339261223142347617901-1
13	0.4998032923765095789488353+0	0.6255333880575099221001711-1
14	0.4998311673467357528099563+0	0.6254550122226665985960952-1
15	0.4998535090524275498334804+0	0.6253927301393207138839993-1
16	0.4998716911400500233692867+0	0.6253424175026022321354261-1
17	0.4998866858891195975515897+0	0.6253011917047419224366962-1
18	0.4998991971771692743253767+0	0.6252669886828711531437449-1
19	0.4999097447421078882977161+0	0.6252382989725239067226655-1
20	0.4999187191160587306248568+0	0.6252139979987882585789658-1
21	0.4999264183447872010118851+0	0.6251932340168467508461636-1
22	0.4999330730828293368672846+0	0.6251753523559262072777206-1
23	0.4999388640880470893800002+0	0.6251598431194931680603091-1
24	0.4999439346373893106544550+0	0.6251463043927422864958548-1
25	0.4999483994818230691187482+0	0.6251344159158641393491053-1
26	0.4999523514007295296044060+0	0.6251239199535311867247599-1

TABLE 16.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 3

[Zero moment = 0.131102877714605990523242O+1]

J	B(J)	G(J)
1	0.599070117367796103719961 2+0	0.000000000000000000000000+0
2	0.496575308391231306594127 9+0	0.980615755213986264643700 1-1
3	0.494206596223643390652613 2+0	0.651230668729688376094031 6-1
4	0.497250818634630970177733 4+0	0.634567464445322460007658 1-1
5	0.498368334706216340082588 0+0	0.630216622043832448774568 3-1
6	0.498920711576231234352178 0+0	0.628286467143426334129480 4-1
7	0.499233280203038929619131 6+0	0.627259574563263164285337 9-1
8	0.499427260731962454268728 4+0	0.626648934740727728321655 6-1
9	0.499555699912365042805876 6+0	0.626256361414139322381913 1-1
10	0.499645578814677471104087 7+0	0.625969070388605559717015 3-1
11	0.499710586535602459880293 7+0	0.625798878934264028037420 4-1
12	0.499759211990766842308577 9+0	0.625658735590522646928755 5-1
13	0.499796532929108761544824 6+0	0.625552496308024478972138 3-1
14	0.499825801331674556284820 9+0	0.625470040773002048207789 3-1
15	0.499849178301213752892490 2+0	0.625404762592293949083724 9-1
16	0.499868145561172878366753 0+0	0.625352200261280080459743 2-1
17	0.499883746577098789819505 0+0	0.625309252461306495897741 1-1
18	0.499896733416477647648937 2+0	0.625273709037046739634447 1-1
19	0.499907659232216016882478 1+0	0.625243960404170770877764 2-1
20	0.499916938222896604637965 8+0	0.625218811808746240241435 1-1
21	0.499924885515476697837062 8+0	0.625197361320447931117869 7-1
22	0.499931744302882749318812 6+0	0.625178917734056964261523 0-1
23	0.499937704687937266467789 7+0	0.625162944126408927416994 5-1
24	0.499942917009168579412187 2+0	0.625149018297775038704097 1-1
25	0.499947501421055273199258 3+0	0.625136804561850343580875 0-1
26	0.499951554885432493096091 2+0	0.625126033310022889335515 7-1

TABLE 17.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 4

[Zero moment = 0.87401918476403993682161 32+0]

J	B(J)	G(J)
1	0.449302586025847077789970 9+0	0.000000000000000000000000+0
2	0.484447295041690906962694 68+0	0.722951330199541333776938 1-1
3	0.495249663949866313596771 0+0	0.643993398113639147117420 3-1
4	0.497531460126648474842410 8+0	0.633384424625148196363055 5-1
5	0.498502398929222602074858 4+0	0.629740414355197899365238 4-1
6	0.498994049681456571648068 1+0	0.628041535738491752267134 6-1
7	0.499277642257626070669268 9+0	0.627118234556996633952594 2-1
8	0.499456112557136534252684 0+0	0.626560069209002523968350 7-1
9	0.499575707494535639763004 7+0	0.626196890647933995384899 4-1
10	0.499659760530890784951766 1+0	0.625947332247569339011627 2-1
11	0.499721086663317039323433 4+0	0.625768467495498796505799 1-1
12	0.499767202348813676971452 3+0	0.625635895720106322290233 3-1
13	0.499802753794764967867817 6+0	0.625534908876194717321129 1-1
14	0.499830738951502700255339 0+0	0.625456210888551924405776 2-1
15	0.499853162724615808129097 5+0	0.625393691583752757941453 3-1
16	0.499871407187828693479933 1+0	0.625343200412626787885352 8-1
17	0.499886450189805331450069 6+0	0.625301837684460437968834 7-1
18	0.499898999389120993778915 3+0	0.625267527890005287735651 1-1
19	0.499909577152177394808255 9+0	0.625238753698978537734296 2-1
20	0.499918575876971254628475 9+0	0.625214385009643013490968 8-1
21	0.499926294958676133347744 3+0	0.625193566116453980319619 6-1
22	0.499932966044148591641050 1+0	0.625175634959765727522142 4-1
23	0.499938770632039640210121 3+0	0.625160093002311462521941 9-1
24	0.499943852559905159413465 0+0	0.625146523220062499408939 6-1
25	0.499948327008262527668916 9+0	0.625134608627690942542901 2-1
26	0.499952287069381990778530 4+0	0.625124090545419969540846 2-1

TABLE 18.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 5

[Zero moment = 0.12143253239437908059099 71+1]

J	B(J)	G(J)
1	0.5773502691896257645091488+0	0.0000000000000000000000000+0
2	0.5073244852932461776984732+0	0.9785159320496508915891917-1
3	0.4948795105676222733388612+0	0.6607151331035211590372786-1
4	0.4972672078130981410627300+0	0.6342595664823952735501140-1
5	0.4983825039421185072184619+0	0.6301773597756086989783538-1
6	0.4989282968380429702512189+0	0.6282606066997644894368888-1
7	0.4992378482546192918920371+0	0.6272450039424414981486837-1
8	0.4994302188104353046369573+0	0.6266398039664500288805171-1
9	0.4995579238210856127905567+0	0.6262502749104794835022006-1
10	0.4996470238734425496081292+0	0.6259848117780434682202012-1
11	0.4997116540121911954372261+0	0.6257957838022860350318375-1
12	0.4997600227601140120027351+0	0.6256564159159946139222956-1
13	0.4997971631227829280281298+0	0.6255507132375737305387282-1
14	0.4998263008275896318487449+0	0.6254686407717625597071578-1
15	0.4998495808796588927641945+0	0.6254036433358142533188712-1
16	0.4998684747575800217293738+0	0.6253512914322700083485744-1
17	0.4998840191953451561375046+0	0.6253085044489080305010964-1
18	0.4998969617132075425616461+0	0.6252730860304818867801960-1
19	0.4999078523170157899616100+0	0.6252434360306568242235083-1
20	0.4999171029802198047639034+0	0.6252183662993729371625628-1
21	0.4999250272262579700092354+0	0.6251969796210921368302859-1
22	0.4999318670726330257024682+0	0.6251785821789478865738821-1
23	0.4999378117472204483070474+0	0.6251626576986311136886451-1
24	0.4999430109281509055388034+0	0.6251487677616924519194694-1
25	0.4999475842651011210207025+0	0.6251365841626843977796821-1
26	0.4999516283295563852513043+0	0.6251258384016639071016247-1

TABLE 19.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 6

[Zero moment = 0.91074399295784310443247 81+0]

J	B(J)	G(J)
1	0.4618802153517006116073190+0	0.0000000000000000000000000+0
2	0.4842922474383111053137346+0	0.7412328435886561499483500-1
3	0.4951303456690923598814542+0	0.6428962495683730122573380-1
4	0.4975060384375699610621943+0	0.6336272305060683443434178-1
5	0.4984901453757904384305800+0	0.6297744324108638830802408-1
6	0.4989871213782979101558020+0	0.6280650351430805483073168-1
7	0.4992734249441331096310103+0	0.6271315883112505743978424-1
8	0.4994533523708811746413716+0	0.6265685484936632039630714-1
9	0.4995738038362481431348757+0	0.6262025937055727058592352-1
10	0.4996583926532631487561469+0	0.6259513511832413969369233-1
11	0.4997200709624524781621921+0	0.6257714052088745618903800-1
12	0.4997664275917414100886705+0	0.6256381077933834168882223-1
13	0.4998021494165093515891021+0	0.6255366159283641267640380-1
14	0.4998302584402616058791617+0	0.6254575556732714490780611-1
15	0.4998527744167714238694441+0	0.6253947697735688558167979-1
16	0.4998710889242144546537165+0	0.6253440780695731512164820-1
17	0.4998861860876615158601096+0	0.6253025616112264421992180-1
18	0.4998987778240754602995538+0	0.6252681319936344490558501-1
19	0.4999093894584215274819010+0	0.6252392630303689392625671-1
20	0.4999184154877501617292742+0	0.6252148184005466529684303-1
21	0.4999261568245507204524516+0	0.6251939379440144269702956-1
22	0.4999328462309934921190331+0	0.6251759608535072589354035-1
23	0.4999386660380116782913450+0	0.6251603726861702687955144-1
24	0.4999437607127858799346190+0	0.6251467681103248427978876-1
25	0.4999482459181128238046202+0	0.6251348242644722062583708-1
26	0.4999522151399067760115839+0	0.6251242814086083723663851-1

TABLE 20.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 7

[Zero moment = 0.116359257121826937530251 8+1]

J	B(J)	G(J)
1	0.56335387900140439567340 88+0	0.000000000000000000000000000+0
2	0.51195837172094485645659 40+0	0.96845969387166064301155 48-1
3	0.49625404673071696960474 99+0	0.66957340493676726819436 45-1
4	0.49725430875851471199814 88+0	0.63467144933885798943236 79-1
5	0.49639213921247103990606 51+0	0.63012138260767852298107 68-1
6	0.49693254894504937802496 49+0	0.62824804237797974647370 65-1
7	0.49924044660006275102497 53+0	0.62723657047411175222969 01-1
8	0.49943190271305641477045 80+0	0.62663461797919091807704 79-1
9	0.49955907593296958286739 62+0	0.62624680995773023120914 88-1
10	0.49964784675963164950967 92+0	0.62596238711117059933756 20-1
11	0.49971226203892259461609 46+0	0.62579402106340346848105 36-1
12	0.49976048467564324034857 43+0	0.62565509448978016164028 07-1
13	0.49979752223505121319547 4+0	0.62554969727168892504778 78-1
14	0.49982658551314196710217 71+0	0.62546784291835231060121 87-1
15	0.49984981036481219799007 67+0	0.62540300536753760513348 07-1
16	0.49986866243948693639494 09+0	0.62535077332643662816793 06-1
17	0.49988417464120915492145 87+0	0.62530807796269686002445 40-1
18	0.49989709190254623710167 82+0	0.62527273077312721981394 51-1
19	0.49990796243812218073340 44+0	0.62524313698309861081003 39-1
20	0.49991719695467873040970 69+0	0.62521811220151772509293 86-1
21	0.49992510806266695407519 17+0	0.62519676189706466981178 00-1
22	0.49993193711024427679461 24+0	0.62517840024335039246419 96-1
23	0.49993787282697783073672 76+0	0.62516249429118253523981 75-1
24	0.49994306451483324664063 50+0	0.62514862482005896020834 21+0
25	0.49994763153591525065465 07+0	0.62513645840695319145866 66-1
26	0.49995167023931209996809 97+0	0.62512572718353702141732 50-1

TABLE 21.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 8

[Zero moment = 0.930874056974615500242014 5+0]

J	B(J)	G(J)
1	0.46946156583450366306117 40+0	0.000000000000000000000000000+0
2	0.48495269501359121365939 96+0	0.75472668470712446415518 46-1
3	0.49487717972208515512340 72+0	0.64230434286440209983939 76-1
4	0.49751757426712366335192 54+0	0.63368847536188492308061 22-1
5	0.49648032285205286291480 41+0	0.62980768343674510932524 75-1
6	0.49696331218798263498025 23+0	0.62807706571374568744761 97-1
7	0.49927098312527078610531 30+0	0.62713947246284079299255 83-1
8	0.49945176273818459296613 90+0	0.62657341769435887562168 90-1
9	0.49957270871420814479204 63+0	0.62620587782617768376572 33-1
10	0.49965760630866299604996 10+0	0.62595366225126329614483 88-1
11	0.49971948744481371032236 88+0	0.62577309342462126543609 75-1
12	0.49976598271783761786740 56+0	0.62563937829381134601595 07-1
13	0.49980180251949190150142 99+0	0.62553759592505719053424 61-1
14	0.49982998273512828789357 33+0	0.62545832740521834081594 35-1
15	0.49985255168118345464774 17+0	0.62539538831727981745348 47-1
16	0.49987090641260788566785 62+0	0.62534458143282897317007 86-1
17	0.49988603466882952445407 84+0	0.62530297670760253716957 91-1
18	0.49989865081771810792906 09+0	0.62526847831267160803793 48-1
19	0.49990928188622573632741 15+0	0.62523955496571682820156 0-1
20	0.49991832357843414793882 49+0	0.62521506676890052538069 93-1
21	0.49992607767902451715214 07+0	0.62519415100099044994296 68-1
22	0.49993277759106960717220 88+0	0.62517614496821801149209 24-1
23	0.49993860612362533986026 70+0	0.62516053290558937310915 56-1
24	0.49994370810547961387774 97+0	0.62514690838312036424912 98-1
25	0.49994819947634745082536 13+0	0.62513494776904218125856 91-1
26	0.49995217393663853864761 69+0	0.62512439071464913669386 26-1

TABLE 22.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 9

[Zero moment = 0.24286506478875816118199 42+1]

J	B(J)	G(J)
1	0.43118492653829842249225 25+0	0.[illegible]+0
2	0.53441126771653426518509 37+0	0.12158949303607898218223 57+0
3	0.49970816633571532451144 00+0	0.63023351816435226003327 03-1
4	0.49999799232365264202166 47+0	0.62507977503130063420355 36-1
5	0.50000007535594648049484 54+0	0.62499961027982358057297 57-1
6	0.50000000132133121427946 95+0	0.62499996622166132044487 69-1
7	0.49999999958680185478657 7+0	0.62500000070427785283640 4-1
8	0.49999999999686242605154 53+0	0.62500000002320577669498 52-1
9	0.50000000000027766450476 4+0	0.62500000000049563493492 2-1
10	0.50000000000001093503149 5+0	0.62499999999981083683095 9-1
11	0.49999999999999998009804 23+0	0.62499999999999870926716 5-1
12	0.49999999999999999888879 28+0	0.62500000000000016722366 0-1
13	0.50000000000000000140255+0	0.62500000000000001995067-1
14	0.50000000000000000116405+0	0.62499999999999999845983-1
15	0.49999999999999999999913+0	0.62499999999999999997285-1
16	0.49999999999999999999988+0	0.62500000000000000000144-1
17	0.50000000000000000000000+0	0.62500000000000000000004-1
18	0.50000000000000000000000+0	0.62500000000000000000000-1
19	0.50000000000000000000000+0	0.62500000000000000000000-1
20	0.50000000000000000000000+0	0.62500000000000000000000-1
21	0.50000000000000000000000+0	0.62500000000000000000000-1
22	0.50000000000000000000000+0	0.62500000000000000000000-1
23	0.50000000000000000000000+0	0.62500000000000000000000-1
24	0.50000000000000000000000+0	0.62500000000000000000000-1
25	0.50000000000000000000000+0	0.62500000000000000000000-1
26	0.50000000000000000000000+0	0.62500000000000000000000-1

TABLE 23.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 10

[Zero moment = 0.52359877559829887307710 72+0]

J	B(J)	G(J)
1	0.53488059585949489125572 18+0	0.0000000000000000000000000+0
2	0.49984137586425812479345 32+0	0.61781328262170206323624 57-1
3	0.49997553125122126127014 37+0	0.62529019937721039840205 83-1
4	0.50000004314907675138774 82+0	0.62500688746932786122423 55-1
5	0.50000024285702978672929 8+0	0.62499975534713937076509 41-1
6	0.50000000065389531879244 5+0	0.62499991898304073847872 2-1
7	0.49999999973974033169836 9+0	0.62500000223418273573169 8-1
8	0.49999999999803701432039 3+0	0.62500000009961886554093 6-1
9	0.50000000000285940367426+0	0.62499999999796306099918 5-1
10	0.50000000000003647314906+0	0.62499999999987621140787 1-1
11	0.49999999999999968448809 7+0	0.62500000000000175890859 0-1
12	0.49999999999999999415849 2+0	0.62500000000000015363951 0-1
13	0.50000000000000000345507+0	0.62499999999999999867722 3-1
14	0.50000000000000000008687+0	0.62499999999999999981055 2-1
15	0.49999999999999999999628+0	0.62500000000000000000645-1
16	0.49999999999999999999988+0	0.62500000000000000000231-1
17	0.50000000000000000000000+0	0.62500000000000000000000-1
18	0.50000000000000000000000+0	0.62500000000000000000000-1
19	0.50000000000000000000000+0	0.62500000000000000000000-1
20	0.50000000000000000000000+0	0.62500000000000000000000-1
21	0.50000000000000000000000+0	0.62500000000000000000000-1
22	0.50000000000000000000000+0	0.62500000000000000000000-1
23	0.50000000000000000000000+0	0.62500000000000000000000-1
24	0.50000000000000000000000+0	0.62500000000000000000000-1
25	0.50000000000000000000000+0	0.62500000000000000000000-1
26	0.50000000000000000000000+0	0.62500000000000000000000-1

TABLE 24.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 11

[Zero moment = 0.26220575542921198104648 40+1]

J	B(J)	G(J)
1	0.76275976350181318806232 60+0	0.00000000000000000000000000+0
2	0.43564062201772584069702 94+0	0.84864209849324680621461 80-1
3	0.48366222538122579575042 04+0	0.60157841661904048857103 78-1
4	0.49263596377578349021290 61+0	0.61936782649710259327191 00-1
5	0.49581606224032412128899 54+0	0.62252225850376234928122 74-1
6	0.49730257536834221985949 69+0	0.62362058866123350617159 78-1
7	0.49811642966365299469104 06+0	0.62412603834396484567358 17-1
8	0.49861023559012693851871 47+0	0.62439862243362402279938 61-1
9	0.49893235219128576501208 67+0	0.62456173524845196085911 06-1
10	0.49915410526386075396124 90+0	0.62466682349204476036653 92-1
11	0.49931326702827978879301 42+0	0.62473837678932023057160 68-1
12	0.49943136816298155768635 45+0	0.62478923545166552066722 95-1
13	0.49952141447585610808434 96+0	0.62482664852124452218389 46-1
14	0.49959164010952059207520 87+0	0.62485495522403585277844 99-1
15	0.49964746473871566551163 76+0	0.62487687955087857133768 16-1
16	0.49969257411422501241471 84+0	0.62489420033416897495573 42-1
17	0.49972954572239930068530 85+0	0.62490811849289424316552 94-1
18	0.49976022599783213572554 99+0	0.62491946817853596456197 47-1
19	0.49978596578791200331263 38+0	0.62492884321769769832292 32-1
20	0.49980777184333151074733 83+0	0.62493667553400574805974 28-1
21	0.49982640690162783702996 17+0	0.62494328529524016541160 95-1
22	0.49984245737869855738922 02+0	0.62494891385259961605674 81-1
23	0.49985638012025178573758 93+0	0.62495374589778387784886 76-1
24	0.49986853530304526270629 22+0	0.62495792468704935279870 46-1
25	0.49987920998487225992165 61+0	0.62496156270292163802328 51-1
26	0.49988863522232244969963 31+0	0.62496474925045507998834 47-1

TABLE 25.- THREE-TERM RECURRENCE RELATION PARAMETERS FOR WEIGHT FUNCTION 12

[Zero moment = 0.21032731579881813917625 29+1]

J	B(J)	G(J)
1	0.71317412781265985500762 91+0	0.00000000000000000000000000+0
2	0.45746497560030029780733 42+0	0.91382663418651905053559 04-1
3	0.48800031686832957892021 67+0	0.62141575465856394574357 41-1
4	0.49444571369213548868741 63+0	0.62562660897512220975928 68-1
5	0.49680597898226656209238 60+0	0.62562998070354659169569 08-1
6	0.49792630104070508970930 34+0	0.62548038918981913896813 32-1
7	0.49854527669645856701963 7+0	0.62536410665558034612965 87-1
8	0.49892313679443215999648 18+0	0.62528210159022223349085 50-1
9	0.49917070460551563172553 53+0	0.62522387682926361766281 02-1
10	0.49934170302375242012127 30+0	0.62518154238053932808264 64-1
11	0.49946475559582878204325 94+0	0.62514997299971785384899 78-1
12	0.49955625446244734358394 19+0	0.62512587603250538684050 53-1
13	0.49962613823219801632490 37+0	0.62510709875903630600991 49-1
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